DERIVATIVES

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DERIVATIVES

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PREFACE

To the Teacher

This is an introductory book on mathematical analysis covering derivatives and their applications. The student should already be familiar with the concept of functions and limits in order to succeed in learning the material in this book. This book is divided into two chapters, structured as follows:

- *Chapter 1 covers differentiation. The first section of this chapter provides a basic introduction to derivatives with the help of tangents and velocity. In the second section we discuss techniques of differentiation which will be vital in the remaining part of the book. The last section discusses further rules of differentiation of elementary functions.*
- *Chapter 2 builds on the material of the previous chapter and covers applications of derivatives. Sections discuss the applications in calculation of limits, analyzing functions, optimization problems and curve plotting in order.*

With the help of effective colors, illustrations and pictures, visualization of the material is effectively improved to aid in students' understanding, especially the examples and the graphs. The book follows a linear approach, with material in the latter sections building on concepts and math covered previously in the text. For this reason there are several self-test 'Check Yourself ' sections that check students' understanding of the material at key points. 'Check Yourself ' sections include a rapid answer key that allows students to measure their own performance and understanding. Successful completion of each self-test section allows students to advance to the next topic. Each section is followed by a number of exercises. Many of the problems reflect skills or problem-solving techniques encountered in the section. All of these problems can be solved using skills the student should already have mastered. Each chapter ends with review materials, beginning with a brief summary of the chapter highlights. Following these highlights is a concept check test that asks the student to summarize the main ideas covered in the chapter. Following the concept check, review tests cover material from the chapter.

Acknowledgements

Many friends and colleagues were of great help in writing this textbook. We would like to thank everybody who helped us at Zambak Publications, especially Mustafa Kırıkçı and Cem Giray. Special thanks also go to Şamil Keskinoğlu and Serdar Çam for their patient typesetting and design.

The Authors

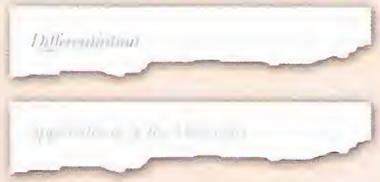
To the Student

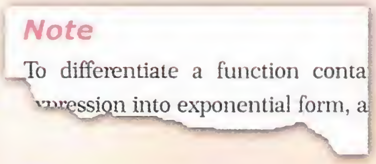
This book is designed so that you can use it effectively. Each chapter has its own special color that you can see at the bottom of the page.

Different pieces of information in this book are useful in different ways. Look at the types of information, and how they appear in the book:

Chapter 1

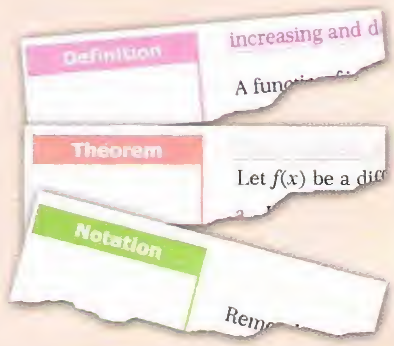
Chapter 2

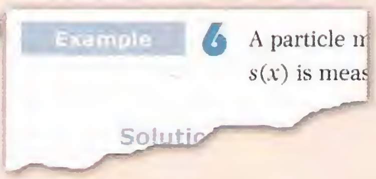




***Notes** help you focus on important details. When you see a note, read it twice! Make sure you understand it.*

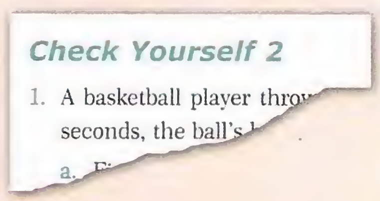
***Definition** boxes give a formal description of a new concept. **Notation** boxes explain the mathematical way of expressing concepts. **Theorem** boxes include propositions that can be proved. The information in these boxes is very important for further understanding and for solving examples.*



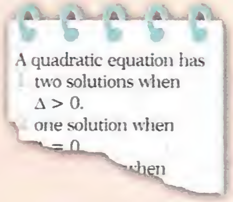


***Examples** include problems and their solution along with an explanation, all of which are related to the topic. The examples are numbered, so you can find them easily in the book.*

***Check Yourself** sections help you check your understanding of what you have just studied. Solve them alone and then compare your answers with the answer key provided. If your answers are correct, you can move on to the next section.*

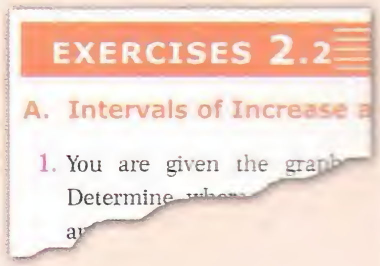


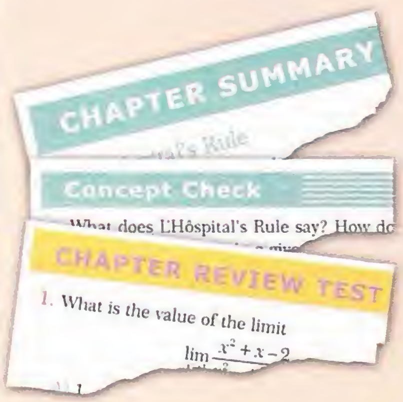
A small notebook in the left or right margin of a page reminds you of material that is related to the topic you are studying. Notebook text helps you to remember the math you need to understand the material. It might help you to see your mistakes, too! Notebooks are the same color as the section you are studying.



Special windows highlight important new information. Windows may contain formulas, properties, or solution procedures, etc. They are the same color as the color of the section.

***Exercises** at the end of each section cover the material in the whole section. You should be able to solve all the problems without any special symbol. (✪) next to a question means the question is a bit more difficult. (✪✪) next to a question means the question is for students who are looking for a challenge! The answers to the exercises are at the back of the book.*

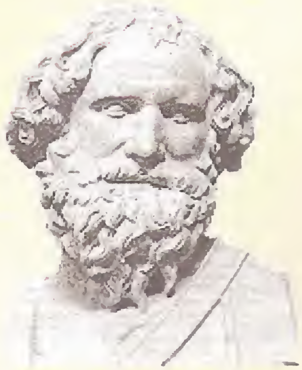




*The **Chapter Summary** summarizes all the important material that has been covered in the chapter. The **Concept Check** section contains oral questions. In order to answer them you don't need a paper or pen. If you answer Concept Check questions correctly, it means you know that topic! The answers to these questions are in the material you studied. Go back over the material if you are not sure about an answer to a Concept Check question. Finally, **Chapter Review Tests** are in increasing order of difficulty and contain multiple choice questions. The answer key for these tests is at the back of the book*

CONTENTS

INTRODUCTION



Everything is changing around us and we need a language to describe this changing world. How fast and in which direction is the change occurring? Derivatives help us to answer these questions by telling the rate of change.

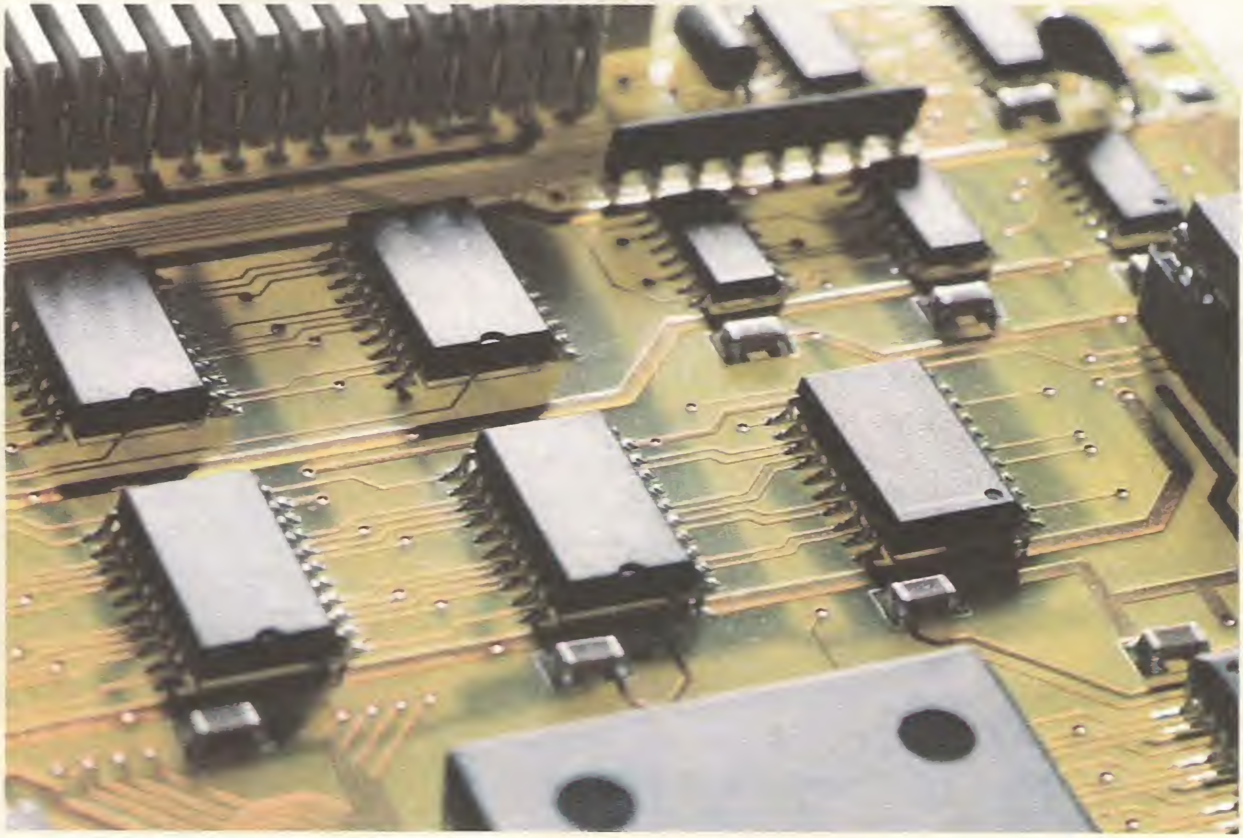
The history of calculus (derivatives) is an interesting story of personalities, intellectual movements, and controversy. Before the time of Greek scientists, Archimedes was concerned with the problem of finding the unique tangent line to a given curve at a given point on the curve. At that time the mathematicians knew how to find the line tangent to a circle using the fact that the tangent line is perpendicular to the radius of circle at any given point. They also discovered how to construct tangent lines to other particular curves such as parabolas, ellipses, and hyperbolas.

The problems of motion and velocity are basic to our understanding of derivatives today. Derivatives originated with Aristotle's (384-322 B.C) study of physics. The problems of motion are closely associated with the ideas of continuity and the infinity (infinitely small and infinitely large magnitudes).

Although calculus has been "a dramatic intellectual struggle which has lasted for 2500 years", the credit for its invention is given to two mathematicians: Gottfried Wilhelm Leibniz (1646-1716) and Isaac Newton (1646-1716). Newton developed calculus first, but he did not publish his ideas for various reasons. This allowed Leibniz to publish his own version of calculus first, in 1684. Fifteen years later, controversy erupted over who should get the credit for the invention of calculus. The debate got so heated that the Royal Society set up a commission to investigate the question. The commission decided in favour of Newton, who happened to be president of the society at the time. The consensus today is that each man invented calculus independently. So both of them deserve the credit as an inventor of calculus.

Many famous mathematicians, such as Joseph Louis Legrange and Augustin Louis Cauchy, went on to study calculus further. Lagrange went on to study a purely algebraic form of the derivative whereas Cauchy went on to find the derivatives of functions, giving the modern definition of derivatives as well as introducing the chain rule, all of which will be dicussed further in detail in this book.

Today, derivatives and its applications in everday life, from physics and chemistry to technology, engineering, economics and other sciences, are one of the most important and most useful areas of mathematics. New applications are always being devised and applied to our modern world.

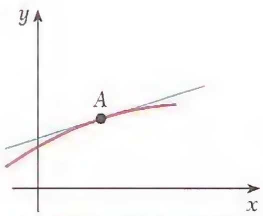


CHAPTER 1

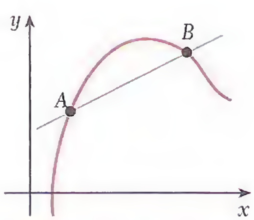
DIFFERENTIATION

1 INTRODUCTION TO DERIVATIVES

A. TANGENTS

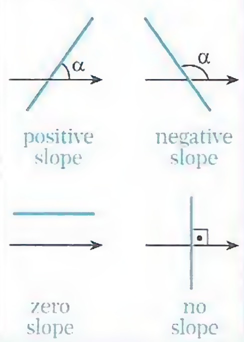


a tangent line to a curve

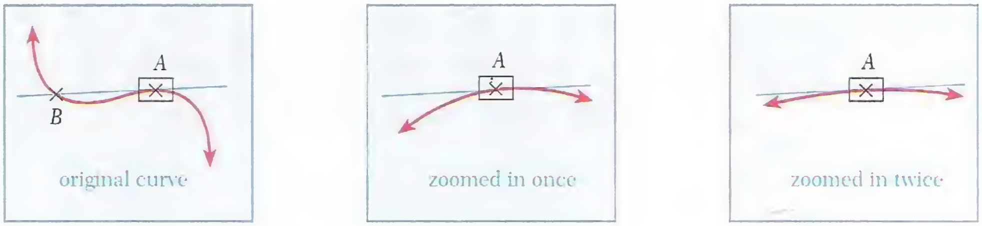


a secant line to a curve

The slope of the line is the tangent of the angle between the line and the positive x-axis.

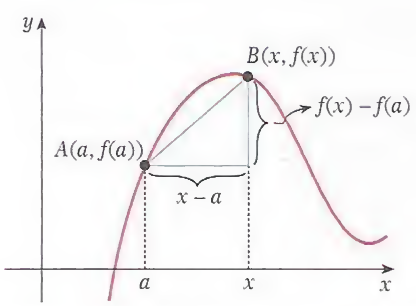


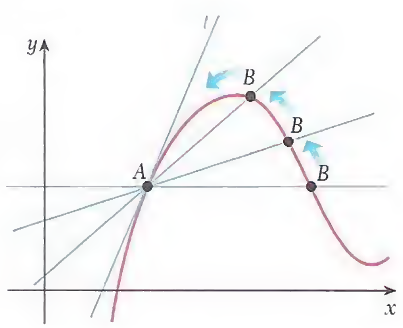
The word 'tangent' comes from the Latin word *tangens*, which means 'touching'. Thus, a tangent line to a curve is a line that "just touches" the curve. In other words, a tangent line should be parallel to the curve at the point of contact. How can we explain this idea clearly? Look at the figures below.



As we zoom in to the curve near the point A, the curve becomes almost indistinguishable from the tangent line. So, the tangent line is parallel to the curve at the point A.

How can we find the equation of a tangent to a curve at a given point? The graphs below show one approach.





The first graph shows the curve $y = f(x)$. The points $A(a, f(a))$ and $B(x, f(x))$ are two points on this curve. The secant line AB has slope m_{AB}, where

$$m_{AB} = \frac{f(x) - f(a)}{x - a}.$$

Now suppose that we want to find the slope of the tangent to the curve at point A. The second graph above shows what happens when we move point B closer and closer to point A on the curve. We can see that the slope of the secant line AB gets closer and closer to the slope of the tangent at A (line t). In other words, if m is the slope of the tangent line, then as B approches A, m_{AB} approaches m.

Definition

tangent line

The **tangent line** to the curve $y = f(x)$ at the point $A(a, f(a))$ is the line through A with the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

Example 1 Find the equation of the tangent line to the curve $y = x^2$ at the point $A(1, 1)$.

Solution We can begin by calculating the slope of the tangent.

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$m = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} (x+1) = 1 + 1 = 2.$$

The equation of a line through the point (x_1, y_1) with slope m: $y - y_1 = m(x - x_1)$.

Now we can write the equation of the tangent at point $(1, 1)$:

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 1)$

$y = 2x - 1.$

Example 2 Find the equation of the tangent line to the curve $y = x^3 - 1$ at the point $(-1, -2)$.

Solution Here we have $a = -1$ and $f(x) = x^3 - 1$, so the slope is

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$$m = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{(x^3 - 1) - ((-1)^3 - 1)}{x + 1} = \lim_{x \to -1} \frac{x^3 + 1}{x + 1}$$

$$m = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{(x+1)}$$

$$m = \lim_{x \to -1} (x^2 - x + 1) = (-1)^2 - (-1) + 1$$

$$m = 3.$$

So the equation of the tangent line at $(-1, -2)$ with slope $m = 3$ is

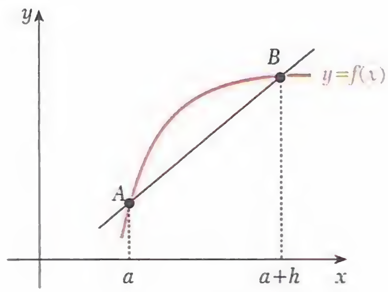
$y - y_1 = m(x - x_1)$

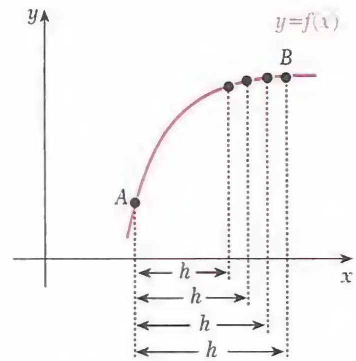
$y - (-2) = 3(x - (-1))$

$y + 2 = 3x + 3$

$y = 3x + 1.$

We can also write the expression for the slope of a tangent line in a different way. Look at the graphs below.





From the first graph, writing $x = a + h$ gives us the slope of the secant line

$$m_{AB} = \frac{f(a+h)-f(a)}{h}.$$

We can see in the second graph that as x approaches a, h approaches zero. So the expression for the slope of the tangent line becomes:

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

THE SLOPE OF A TANGENT LINE TO A CURVE

The slope of a tangent line to a curve $y = f(x)$ at $x = a$ is

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Example 3 Find the equation of the tangent line to the curve $y = x^3$ at the point $(-1, -1)$.

Solution Let $f(x) = x^3$. Then the slope of the tangent at $(-1, -1)$ is

$$m = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{(-1+h)^3-(-1)^3}{h}$$

$(x+y)^3=x^3+3x^2y+3xy^2+y^3$

$$m = \lim_{h\to 0}\frac{(-1)^3+3(-1)^2h+3(-1)h^2+h^3-(-1)^3}{h}$$

$$m = \lim_{h\to 0}\frac{h(3-3h+h^2)}{h} = \lim_{h\to 0}(3-3h+h^2) = 3.$$

So, the equation of the tangent at point $(-1, -1)$ is

$y - (-1) = 3(x - (-1))$

$y + 1 = 3x + 3$

$y = 3x + 2.$

Example 4 Find the equation of the normal line to the curve $y = \frac{2}{x}$ at the point (2, 1).

Solution Recall that a normal line is a line which is perpendicular to a tangent. The product of the slopes m_t of a tangent and m_n of a normal is -1.

Let us begin by finding the slope of the tangent.

$$m_t = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{\frac{2}{2+h}-\frac{2}{2}}{h} = \lim_{h\to0}\frac{\frac{2}{2+h}-1}{h}$$

$$m_t = \lim_{h\to0}\frac{2-(2+h)}{h(h+2)} = \lim_{h\to0}\frac{-h}{h(h+2)} = \lim_{h\to0}\frac{-1}{2+h}$$

$$m_t = -\frac{1}{2}.$$

$m_n \cdot m_t = -1$
The product of slopes of the tangent line and the normal line at a point equals -1.

We have $m_t \cdot m_n = -1$.

So, $m_n = \frac{-1}{m_t} = \frac{-1}{-\frac{1}{2}} = 2.$

The equation of the normal line passing through the point (2, 1) with the slope $m_n = 2$ is

$y - y_1 = m_n(x - x_1)$

$y - 1 = 2(x - 2)$

$y = 2x - 3.$

Check Yourself 1

1. Find the equation of the tangent line to each curve at the given point P.

   a. $f(x) = x^2 - 1$ $\quad P(-1, 0)$

   b. $f(x) = x^3 + 1$ $\quad P(0, 1)$

   c. $f(x) = \frac{1}{x}$ $\quad P(\frac{1}{2}, 2)$

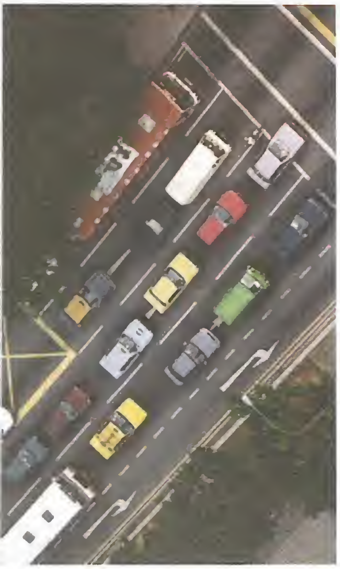
2. Find the equation of the normal line at point P for each curve in the previous question.

Answers

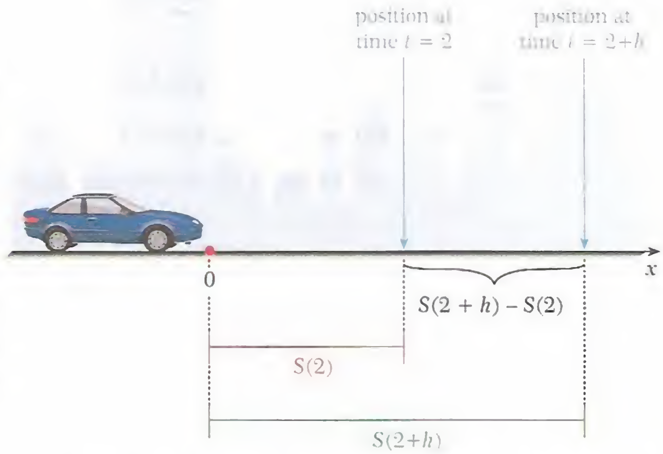
1. a. $y = -2x - 2$ b. $y = 1$ c. $y = -4x + 4$

2. a. $y = \frac{1}{2}x + \frac{1}{2}$ b. $x = 0$ c. $y = \frac{1}{4}x + \frac{15}{8}$

B. VELOCITIES



Imagine you are in a car driving across a city. The velocity of the car will not be constant. Sometimes the car will travel faster, and sometimes it will travel slower.



However, the car has a definite velocity at each moment. This is called the instantaneous velocity of the car. How can we calculate the instantaneous velocity?

To answer this question, let us look at a simpler example: the motion of an object falling through the air. Let $g = 9.8$ m/s^2 be the acceleration of the object due to gravity. We know from physics that after t seconds, the distance that the object will have fallen is

$$s(t) = \frac{1}{2}gt^2 \text{ meters} \quad \text{or} \quad s(t) = 4.9\,t^2 \text{ meters.}$$

Suppose we wish to calculate the velocity of the object after two seconds. We can begin by calculating the average velocity over the time interval $[2, 2 + h]$:

$$\text{average velocity} = \frac{\text{distance travelled}}{\text{elapsed time}}$$

$$= \frac{s(2+h) - s(2)}{h} = \frac{4.9(4 + 4h + h^2 - 4)}{h} = 19.6 + 4.9h$$

If we shorten the time period, the average velocity is becoming closer to 19.6 m/s, the value of instantaneous velocity.

More generally, we can calculate the instantaneous velocity $V(a)$ of an object at time $t = a$ by the limit of the average velocities:

$$V(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

This is not the first time we see the above formula. It is the same formula that we use for the slope of the tangent line to a curve. Remember that

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at $A(a, s(a))$.

Example 5 A stone is dropped from the top of the Eiffel Tower. What is the velocity of the stone after five seconds?

Solution We use the equation of motion $s(t) = 4.9t^2$ to find the velocity V after five seconds:

$$V(5) = \lim_{h\to 0}\frac{s(5+h)-s(5)}{h} = \lim_{h\to 0}\frac{4.9(5+h)^2 - 4.9(5)^2}{h}$$

$$V(5) = \lim_{h\to 0}\frac{4.9(25+10h+h^2-25)}{h} = \lim_{h\to 0}\frac{4.9(10h+h^2)}{h}$$

$$V(5) = \lim_{h\to 0}\frac{4.9h(10+h)}{h} = \lim_{h\to 0}(49+4.9h) = 49 \text{ m/s}.$$

Example 6 A particle moves along a straight line with the equation of motion $s(t) = t^2 + 3t + 1$, where $s(t)$ is measured in meters and t is in seconds.

a. Find the average velocity over the interval [1, 2].

b. Find the instantaneous velocity at $t = 2$.

Solution a. Average velocity is the ratio of distance travelled to elapsed time. So, we have

$$\text{average velocity} = \frac{s(2)-s(1)}{2-1} = \frac{(2^2+3\cdot 2+1)-(1^2+3\cdot 1+1)}{1} = 6 \text{ m/s}.$$

b. Let $V(2)$ be the velocity after two seconds.

$$V(2) = \lim_{h\to 0}\frac{s(2+h)-s(2)}{h} = \lim_{h\to 0}\frac{(h+2)^2+3\cdot(h+2)+1-[2^2+3\cdot 2+1]}{h}$$

$$V(2) = \lim_{h\to 0}\frac{h^2+4h+4+3h+6+1-11}{h} = \lim_{h\to 0}\frac{h^2+7h}{h} = \lim_{h\to 0}(h+7) = 7 \text{ m/s}.$$

Check Yourself 2

1. A basketball player throws a ball upward at a speed of 20 m/s. This means that after t seconds, the ball's height will be $s(t) = 20t - 4.9t^2$.
   a. Find the average velocity of the ball over the interval [1, 2].
   b. Find the instantaneous velocity of the ball after two seconds.
2. The displacement of a particle moving in a straight line is given by the equation of motion $s(t) = 2t^3 + 3t - 2$, where t is measured in seconds and $s(t)$ is in meters.
   a. Find the average velocity of the particle over the following intervals.
      i. [1, 3] ii. [1, 4] iii. [2, 4]
   b. Find the instantaneous velocity of the particle at each time.
      i. $t = 2$ ii. $t = 3$ iii. $t = 4$

Answers

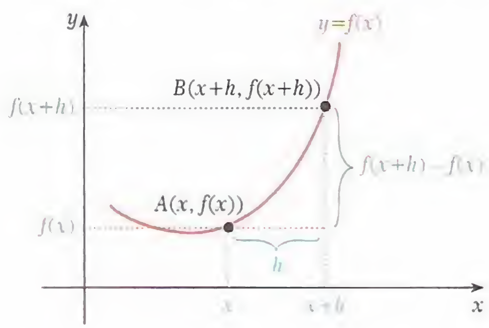
1. a. 5.3 m/s b. 0.4 m/s 2. a. i. 29 m/s ii. 45 m/s iii. 59 m/s b. i. 27 m/s ii. 57 m/s iii. 99 m/s

C. RATES OF CHANGE

In section A we learned how to find the slope of a tangent line and in section B we learned how to calculate the instantaneous velocity of an object from a given acceleration. We can say that acceleration is a **rate of change**: it shows how fast or slowly a quantity (the velocity) changes from one moment to the next. Other examples of rates of change are how fast a population grows, or how fast the temperature of a room changes over time.

The problem of finding a rate of change is mathematically equivalent to finding the slope of a tangent line to a curve. To understand why, suppose y is a quantity that depends on another quantity x. Thus, y is the function of x and we write $y = f(x)$.

Look at the graph of $f(x)$. If x increases by an amount h, then y increases by $f(x + h) - f(x)$.



$f(x+h) - f(x)$ is the change in y that corresponds to a change h in x.

The difference quotient $\dfrac{f(x+h)-f(x)}{h}$ is called the **average rate of change** of y with respect to x over the interval $[x, x + h]$ and can be interpreted as the slope of the secant line AB. If we take the limit of the average rate of change, then we obtain the **instantaneous rate of change** of y with respect to x, which is interpreted as the slope of the tangent line to the curve $y = f(x)$ at $A(x, f(x))$.

The following summarizes this part:

RATES OF CHANGE

1. The **average rate of change** of f over an interval $[x, x + h]$ is
$$\frac{f(x+h)-f(x)}{h}.$$
2. The **instantaneous rate of change** of $f(x)$ at a point x is
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Example 7 A student begins measuring the air temperature in a room at eight o'clock in the morning. She finds that the temperature is given by the function $f(t)=16+\frac{2}{3}t^2$ °C, where t is in hours. How fast was the temperature rising at 11:00?

Solution We are being asked to find the instantenous rate of change of the temperature at $t = 3$, so we need to find the following limit:

$$\begin{aligned}
\text{rate of change} &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\
&= \lim_{h\to 0}\frac{16+\frac{2}{3}(3+h)^2-(16+\frac{2}{3}(3)^2)}{h}\\
&= \lim_{h\to 0}\frac{16+\frac{2}{3}\cdot(9+6h+h^2)-16-\frac{2}{3}\cdot 9}{h}\\
&= \lim_{h\to 0}\frac{\frac{2}{3}(9+6h+h^2-9)}{h}\\
&= \lim_{h\to 0}\frac{2h(6+h)}{3h}\\
&= \frac{2}{3}\lim_{h\to 0}(6+h)\\
&= \frac{2}{3}\cdot 6 = 4 \text{ °C per hour}
\end{aligned}$$

Example 8 A manufacturer estimates that when he produces x units of a certain commodity, he earns $R(x) = x^2 - 3x - 1$ thousand dollars. At what rate is the revenue changing when the manufacturer produces 3 units?

Solution We need to find the instantaneous rate of change of the revenue at $x = 3$, so

$$\text{rate of change} = \lim_{h\to 0}\frac{R(3+h)-R(3)}{h}=\frac{(3+h)^2-3(3+h)-1-(3^2-3\cdot 3-1)}{h}$$

$$= \lim_{h\to 0}\frac{9+6h+h^2-9-3h}{h} = \lim_{h\to 0}\frac{3h+h^2}{h} = \lim_{h\to 0}(3+h) = 3.$$

It follows that revenue is changing at the rate of $3000 per unit when 3 units are produced.

In conclusion, rates of change can be interpreted as the slope of a tangent. Whenever we solve a problem involving tangent lines, we are not only solving a problem in geometry but also solving a great variety of problems in science.

D. DERIVATIVE OF A FUNCTION

Up to now we have treated the expression $\dfrac{f(x+h)-f(x)}{h}$ as a 'difference quotient' of the function $f(x)$. We have calculated the limit of a difference quotient as h approaches zero. Since this type of limit occurs so widely, it is given a special name and notation.

Definition

derivative of a function

The **derivative** of the function $f(x)$ with respect to x is the function $\boldsymbol{f'(x)}$ (read as "f prime of x") defined by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

The process of calculating the derivative is called **differentiation**. We say that $f(x)$ is **differentiable** at c if $f'(c)$ exists.

Thus, the derivative of a function $f(x)$ is the function $f'(x)$, which gives

1. the slope of the tangent line to the graph of $f(x)$ at any point $(x, f(x))$,
2. the rate of change of $f(x)$ at x.

FOUR-STEP PROCESS FOR FINDING f′(x)

1. Compute $f(x+h)$.
2. Form the difference $f(x+h)-f(x)$.
3. Form the quotient $\dfrac{f(x+h)-f(x)}{h}$.
4. Compute $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Example 9 Find the derivative of the function $f(x) = x^2$.

Solution To find $f'(x)$, we use the four-step process:

1. $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$
2. $f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$
3. $\dfrac{f(x+h)-f(x)}{h}=\dfrac{2xh+h^2}{h}=\dfrac{h(2x+h)}{h}=2x+h$
4. $\lim\limits_{h\to 0}(2x+h)=2x$

Thus, $f'(x) = 2x$.

Example 10 Find the derivative of the function $f(x) = x^2 - 8x + 9$ at $x = 1$.

Solution We apply the four-step process:

1. $f(x + h) = (x + h)^2 - 8(x + h) + 9 = x^2 + 2xh + h^2 - 8x - 8h + 9$
2. $f(x + h) - f(x) = x^2 + 2xh + h^2 - 8x - 8h + 9 - (x^2 - 8x + 9) = 2xh + h^2 - 8h$
3. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h^2+2xh-8h}{h} = h+2x-8$
4. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(h+2x-8) = 2x-8$

So, $f'(x) = 2x - 8$ and $f'(1) = 2 \cdot 1 - 8 = -6$.

This result tells us that the slope of the tangent line to the graph of $f(x)$ at the point $x = 1$ is -6. It also tells us that the function $f(x)$ is changing at the rate of -6 units per unit change in x at $x = 1$.

Example 11 Let $f(x) = \dfrac{1}{x}$.

a. Find $f'(x)$.

b. Find the equation of the tangent line to the graph of $f(x)$ at the point $(1, 1)$.

Solution a. $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h} = \lim\limits_{h\to 0}\dfrac{-\dfrac{h}{x(x+h)}}{h} = \lim\limits_{h\to 0}\left(-\dfrac{1}{x(x+h)}\right) = -\dfrac{1}{x^2}.$

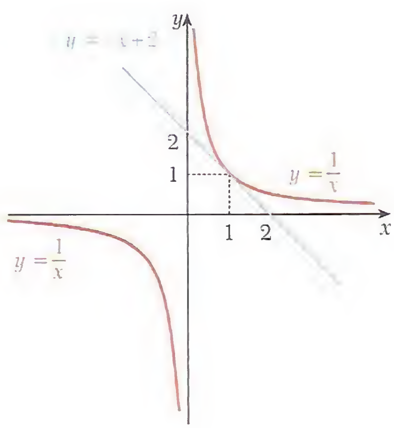
b. In order to find the equation of a tangent line, we have to find its slope and one point on the tangent line. We know that the derivative gives us the slope of the tangent. Let m be the slope of the tangent line, then

$$m = f'(1) = -\frac{1}{1^2} = -1.$$

So, the equation of the tangent line to the graph of $f(x)$ at the point $(1, 1)$ with the slope $m = -1$ is

$y - 1 = -1(x - 1)$

$y = -x + 2.$



Example 12 The function $f(x) = \sqrt{x}$ is given. Find the derivative of $f(x)$ and the equation of the normal line to $f(x)$ at the point $x = 1$.

Solution

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$f'(x) = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Remember that if m_t is the slope of a tangent and m_n is the slope of a normal at the same point, then $m_t \cdot m_n = -1$. So, we can find the slope of the normal from the slope of the tangent. Then we can write the equation of normal line to $f(x)$ at the point $x = 1$.

The slope of the tangent is

$$m_t = f'(1) = \frac{1}{2\cdot\sqrt{1}} = \frac{1}{2}.$$

The slope of the normal is

$$m_n = -\frac{1}{m_t} = -\frac{1}{\frac{1}{2}} = -2.$$

The equation of the normal line is

$y - y_0 = m_n \cdot (x - x_0)$

$y - 1 = -2(x - 1)$ *(Note that $y_0 = f(x_0)$, that is $y_0 = f(1) = 1$)*

$y = -2x + 3.$

Check Yourself 3

1. Find the derivative of the function $f(x) = 2x + 7$.
2. Let $f(x) = 2x^2 - 3x$.
   a. Find $f'(x)$.
   b. Find the equation of the tangent line to the graph of $f(x)$ at the point $x = 2$.
3. Find the derivative of the function $f(x) = x^3 - x$.
4. If $f(x) = \dfrac{1}{\sqrt{x+2}}$, find the derivative of $f(x)$.

Answers

1. 2 2. a. $4x - 3$ b. $y = 5x - 8$ 3. $3x^2 - 1$ 4. $-\dfrac{1}{2\sqrt{(x+2)^3}}$

E. LEFT-HAND AND RIGHT-HAND DERIVATIVES

When we were studying limits we learned that the limit of a function exists if and only if the left-hand and the right-hand limits exist and are equal. Otherwise the function has no limit. From this point, we may conclude that the derivative of a function $f(x)$ exists if and only if

$f'(x^-) = \lim\limits_{h \to 0^-} \dfrac{f(x+h) - f(x)}{h}$ and $f'(x^+) = \lim\limits_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h}$ exist and are equal.

These expressions are respectively called the **left-hand derivative** and the **right-hand derivative** of the function.

Example 13 Show that the function $f(x) = \sqrt{x}$ does not have a derivative at the point $x = 0$.

Solution Here we should find the left-hand derivative and the right-hand derivative. If they exist, then we will check whether they are equal or not.

Let us find the left-hand derivative:

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \to 0} \frac{\sqrt{h}}{h}.$$

Since $h < 0$, $\sqrt{h}$ is undefined and this limit does not exist. So the left-hand derivative does not exist either.

Thus, the function $f(x) = \sqrt{x}$ has no derivative at the point $x = 0$.

Example 14 $f(x)$ is given as $f(x) = \begin{cases} x^2 - 1, & x \geq 1 \\ 2x - 2, & x < 1 \end{cases}$.

Does this function have a derivative at the point $x = 1$?

Solution We will find the left-hand and the right-hand derivatives.

$$f'(1^-) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(1+h) - 2 - 0}{h} = \lim_{h \to 0^-} \frac{2h}{h} = 2$$

$$f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1 - 0}{h} = \lim_{h \to 0^+} \frac{h^2 + 2h}{h} = \lim_{h \to 0^+} (h+2) = 2.$$

The left-hand and the right-hand derivatives are equal to each other. Thus, the derivative of the function at the point $x = 1$ exists and

$$f'(1) = f'(1^-) = f'(1^+) = 2.$$

F. DIFFERENTIABILITY AND CONTINUITY

Recall that if $f'(c)$ exists, then the function $f(x)$ is differentiable at point c. Similarly, if $f(x)$ is differentiable on an open interval (a, b), then it is differentiable at every number in the interval (a, b).

Example 15 Where is the function $f(x) = |x|$ differentiable?

Solution We can approach this problem by testing the differentiability on three intervals: $x > 0$, $x < 0$ and $x = 0$.

1. If $x > 0$, then $x + h > 0$ and $|x + h| = x + h$.

   Therefore, for $x > 0$ we have

   $$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

   So, $f'(x)$ exists and $f(x)$ is differentiable for any $x > 0$.

2. If $x < 0$, then $|x| = -x$ and $|x + h| = -(x + h)$ if we choose h small enough such that it is nearly equal to zero.

   Therefore, for $x < 0$ we have

   $$f'(x) = \lim_{x \to 0} \frac{|x+h| - |x|}{h} = \lim_{x \to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

   So, $f'(x)$ exists and $f(x)$ is differentiable for any $x < 0$.

3. For $x = 0$ we have to investigate the left-hand and the right-hand derivatives separately:

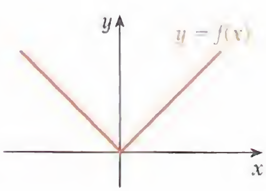
   $$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

   $$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

   Since these limits are different, $f'(x)$ does not exist. So, $f(x)$ is not differentiable for $x = 0$.

In conclusion, $f(x)$ is differentiable for all the values of x except 0.

Alternatively, from the graph of $f(x)$, we can see that $f(x)$ does not have a tangent line at the point $x = 0$. So, the derivative does not exist.



Note that the function does not have a derivative at the point where the graph has a 'corner'.

A function f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

If a function $f(x)$ is differentiable at a point, then its graph has a non-vertical tangent line at this point. It means that the graph of the function cannot have a 'hole' or 'gap' at this point. Thus, the function must be continuous at this point where it is differentiable.

Note

If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

The converse, however, is not true: a continuous function may not be differentiable at every point.

For example, the function $f(x) = |x|$ is continuous at 0, because $\lim_{x \to 0} f(x) = 0 = f(0)$.

But it is not differentiable at the point $x = 0$.

Example 16 The piecewise function $f(x)$ is given as $f(x) = \begin{cases} 2x^2 - x, & x > 2 \\ 6, & x = 2. \\ x^3 - 2, & x < 2 \end{cases}$

a. Is $f(x)$ continuous at $x = 2$?

b. Is $f(x)$ differentiable at $x = 2$?

Solution a. Since $\lim_{x \to 2} f(x) = f(2)$, $f(x)$ is continuous at $x = 2$.

b. Let us find the left-hand and the right-hand derivatives of the function $f(x)$ at the point $x = 2$.

$$f(2^-) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2+h)^3 - 2 - 6}{h}$$

$$= \lim_{h \to 0^-} \frac{2^3 + 3 \cdot 2^2 h + 3 \cdot 2h^2 + h^3 - 8}{h} = \lim_{h \to 0^-} \frac{h(12 + 6h + h^2)}{h}$$

$$= \lim_{h \to 0^-} (12 + 6h + h^2) = 12$$

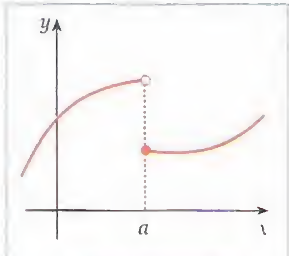
$$f(2^+) = \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{2 \cdot (2+h)^2 - (2+h) - 6}{h}$$

$$= \lim_{h \to 0^+} \frac{2 \cdot (4 + 4h + h^2) - 2 - h - 6}{h} = \lim_{h \to 0^+} \frac{8 + 8h + 2h^2 - 2 - h - 6}{h}$$

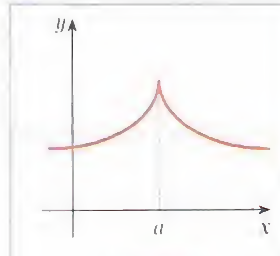
$$= \lim_{h \to 0^+} \frac{h(7 + 2h)}{h} = \lim_{h \to 0^+} (7 + 2h) = 7.$$

Since $f'(2^+) \neq f'(2^-)$, the derivative of the function $f(x)$ does not exist at the point $x = 2$. So, the function is continuous at $x = 2$, but it is not differentiable at the same point.

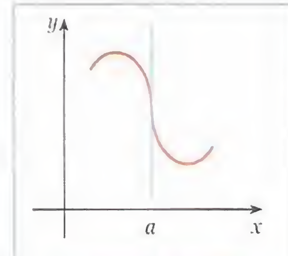
We have seen that a function $f(x)$ is not differentiable at a point if its graph is not continuous at $x = a$. The figures below show two more cases in which $f(x)$ is not differentiable at $x = a$:



a discontinuity



a corner



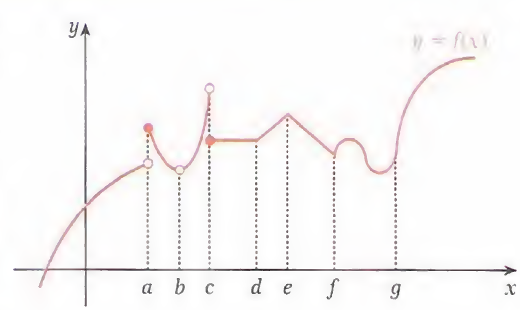
a vertical tangent

CRITERIA FOR DIFFERENTIABILITY

For the following cases the function is not differentiable at a given point:

1. the graph has a discontinuity at the point,
2. the graph has a 'corner' at the point,
3. the graph has a vertical tangent line at the point.

Example 17 Explain why the function shown in the graph on the right is not differentiable at each of the points $x = a, b, c, d, e, f, g$.



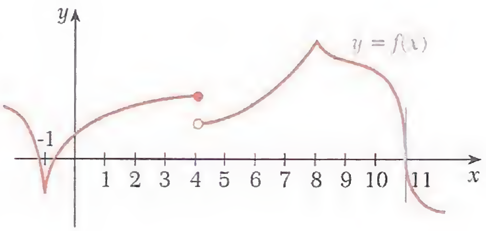
Solution The function $f(x)$ is not differentiable at the points $x = a, b, c$ because it is discontinuous at each of these points. The derivative of the function $f(x)$ does not exist at $x = d, e, f$ because it has a corner at each of these points. Finally, the function is not differentiable at $x = g$ because the tangent line is vertical at that point.

Check Yourself 4

1. Given that $f(x)=\begin{cases} x-1, & x<1 \\ x^2-1, & x\geq 1 \end{cases}$, show that the derivative of $f(x)$ does not exist at the point $x = 1$.

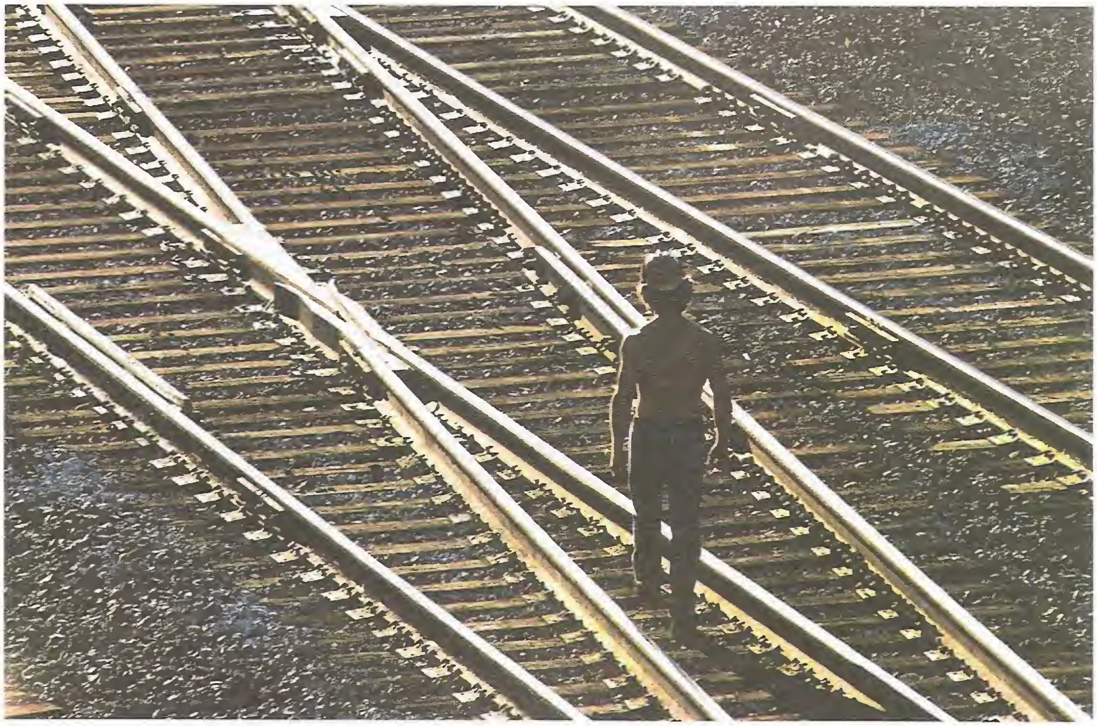
2. $f(x) = |x^2 - 4x + 3|$ is given. Find the derivative of $f(x)$ at the point $x = 3$.

3. The graph of a function f is given below. State, with reasons, the values at which f is not differentiable.



Answers

1. compare $f'(1^-)$ and $f'(1^+)$.
2. does not exist.
3. $x = -1$, corner; $x = 4$, discontinuity; $x = 8$, corner; $x = 11$, vertical tangent.



LINEAR APPROXIMATION

At the beginning of our study of derivatives we have learned that a curve lies very close to its tangent line near the point of contact. This means that for the same value of x near the point of tangency, the values of y on the curve and tangent line are approximately equal to each other. This fact gives us a useful method for finding approximate values of functions.

We can use the tangent line at $(a, f(a))$ as an approximation to the function $f(x)$ when x is near a. The equation of this tangent line is

$$y = f(a) + f'(a)(x - a).$$

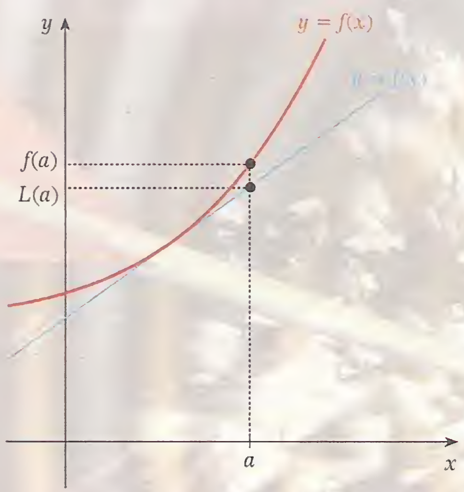
So, our approximation becomes

$$f(x) \approx f(a) + f'(a)(x - a).$$

This type of approximation is called the **linear approximation** or **tangent line approximation** of $f(x)$ at a. The linear function whose graph is the tangent line

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of $f(x)$ at a. The geometric interpretation of linear approximation is show in the figure.



The linear approximation $f(x) \approx L(x)$ is a good approximation when x is near a. It is very useful in physics for simplifying a calculation or a theory. You might think that a calculator can give us better approximation than the linear approximation. But a linear approximation gives an approximation over an entire interval, which can be more useful. For this reason, scientists frequently use linear approximation in their work. The following example illustrate the use of linear approximation method to simplify calculation.

For example, let us find the linearization of the function $f(x)=\sqrt{x+2}$ at $a = 2$, and use it to approximate the numbers $\sqrt{3{,}99}$ and $\sqrt{4{,}01}$.

First, we have to find $f'(2)$, the slope of the tangent line to the curve $f(x)=\sqrt{x+2}$ when $x = 2$. The derivative of $f(x)$ is $f'(x)=(\sqrt{x+2})'=\dfrac{1}{2\sqrt{x+2}}$.

So, $f'(2)=\dfrac{1}{2\sqrt{2+2}}=\dfrac{1}{4}$.

The linearization is given by

$L(x) = f(a) + f'(a) \cdot (x - a)$

$L(x) = f(2) + f'(2) \cdot (x - 2) = 2 + \dfrac{1}{4}(x - 2) = \dfrac{x}{4}+\dfrac{3}{2}$.

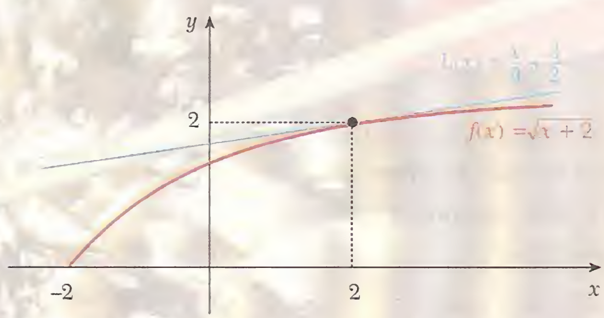
The linear approximation is therefore $\sqrt{x+2} \approx L(x)=\dfrac{x}{4}+\dfrac{3}{2}$.

In particular we have

$$\sqrt{3.99}=\sqrt{1.99+2} \approx L(1.99)=\frac{1.99}{4}+\frac{3}{2}=1.9975$$

$$\sqrt{4.01} \approx L(2.01)=\frac{2.01}{4}+\frac{3}{2}=2.0025.$$

The graphs of $f(x)=\sqrt{x+2}$ and its linear approximation $L(x)=\dfrac{x}{4}+\dfrac{3}{2}$ are shown below. We see that our approximations are overestimates because the tangent line lies above the curve.



The following table shows estimates from the linear approximation with the actual values.

	approximation	actual value
$\sqrt{3.99}$	1.9975	1.99749...
$\sqrt{4.01}$	2.0025	1.00249...

EXERCISES 1.1

A. Tangents

1. Find the slope of the tangent line to the graph of each function at the given point.

   a. $f(x) = 5x - 1$; $x = 3$

   b. $f(x) = 4 - 7x$; $x = 2$

   c. $f(x) = x^2 - 1$; $x = -1$

   d. $f(x) = 3x^2 - 2x - 5$; $x = 0$

   e. $f(x) = x^3 - 3x + 5$; $x = 1$

   f. $f(x) = x + \sqrt{x}$; $x = 4$

   g. $f(x) = \dfrac{1}{x^2}$; $x = 2$

   h. $f(x) = \dfrac{4x}{x+1}$; $x = 2$

2. Find the equation of the tangent line to each function at the given point.

   a. $f(x) = 2x + 5$ at $(2, 9)$

   b. $f(x) = x^2 + x + 1$ at $(1, 3)$

   c. $f(x) = x^3 - x$ at $(2, 6)$

   d. $f(x) = 2\sqrt{x}$ at $(4, 4)$

B. Velocities

3. A particle moves along a straight line with the equation of motion $s(t) = t^2 - 6t - 5$, where s is measured in meters and t is in seconds. Find the velocity of the particle when $t = 2$.

4. If a stone is dropped from a height of 100 m, its height in meters after t seconds is given by $s(t) = 100 - 5t^2$. Find the stone's average velocity over the period $[2, 4]$ and its instantaneous velocity at time $t = 4$.

C. Rates of Change

5. The volume of a spherical cancer tumor is given by the function $V(r) = \frac{4}{3}\pi r^3$, where r is the radius of the tumor in centimeters. Find the rate of change in the volume of the tumor when $r = \frac{2}{3}$ cm.

6. A certain species of eagle faces extinction. After a conservation project begins, it is hoped that the eagle population will grow according to the rule $N(t) = 2t^2 + t + 100$ $(0 \le t \le 10)$, where $N(t)$ denotes the population at the end of the year t. Find the rate of growth of the eagle population when $t = 2$ and the average rate of growth over the interval $[2, 3]$.

7. The fuel consumption (measured in litres per hour) of a car travelling at a speed of v kilometers per hour is $c = f(v)$.

   a. What is the meaning of $f'(v)$?

   b. What does the statement $f'(20) = -0.05$ mean?

D. Derivative of a Function

8. Each limit below represents the derivative of a function $f(x)$ at $x = a$. Find the function f and the number a in each case.

a. $\lim\limits_{h \to 0} \dfrac{(1+h)^{10} - 1}{h}$ b. $\lim\limits_{h \to 0} \dfrac{\sqrt[3]{8+h} - 2}{h}$

c. $\lim\limits_{x \to 4} \dfrac{3^x - 81}{x-4}$ d. $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$

9. Find the derivative of each function.

a. $f(x) = 3 - 2x + x^2$ b. $f(x) = \dfrac{2x+1}{x-1}$

c. $f(x) = \dfrac{3}{\sqrt{x}}$ d. $f(x) = \sqrt{3x+1}$

E. Left-Hand and Right-Hand Derivatives

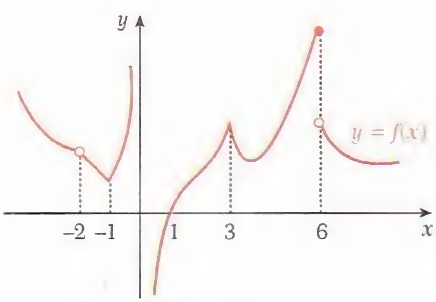
10. Let $f(x) = \begin{cases} 6x, & 0 \le x \le 8 \\ 9x - 24, & 8 < x \end{cases}$.

Does the function have derivative at $x = 8$? Why or why not?

11. Given that $f(x) = |x - 1|$, find $f'(1)$.

F. Differentiability and Continuity

12.



The graph of $f(x)$ is given. At what numbers is $f(x)$ not differentiable? Why?

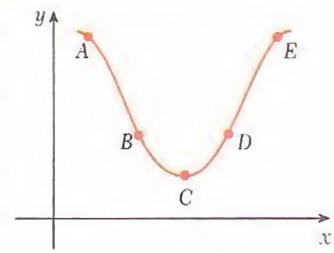
13. Let $f(x) = \begin{cases} x^2 + 7x, & x \le 1 \\ 9x - 24, & x > 1 \end{cases}$.

Does the function have derivative at $x = 1$? Why or why not?

14. Given that $f(x) = \begin{cases} x+6, & x > 3 \\ x^2, & x = 3 \\ x^3 - 6x, & x > 3 \end{cases}$ find $f'(3)$.

Mixed Problems

15. Consider the slopes of the tangent lines to the given curve at each of the five points shown. List these five slopes in decreasing order.



16. If the tangent to the graph of $f(x) = x^2 - 2ax + 3$ at $x = -1$ is parallel to the line $2x - y = 1$, find a.

17. At which point of the curve $y = x^2 + 4$ does its tangent line pass through the origin?

18. An arrow is shot upward on a planet. Its height (in meters) after t seconds is given by $h(t) = 60t - 0.6t^2$.

a. At what time will the arrow reach the top?

b. With what velocity will the arrow hit the ground?

19. Given the continuous function

$$f(x) = \begin{cases} x^2 + 10x + 8, & x \le -2 \\ ax^2 + bx + c, & -2 < x < 0, \\ x^2 + 2x, & x \ge 0 \end{cases}$$

find a, b and c such that its graph has a tangent touching it at three points..

20. Given that $f(x) = |x^2 - 2x|$, find $f'(1)$.

21. Using linear approximation calculate $\sqrt{99}$.

2 TECHNIQUES OF DIFFERENTIATION

A. BASIC DIFFERENTIATION RULES

Up to now, we have calculated the derivatives of functions by using the definition of the derivative as the limit of a difference quotient. This method works, but it is slow even for quite simple functions. Clearly we need a simpler, quicker method. In this section, we begin to develop methods that greatly simplify the process of differentiation. From now on, we will use the notation $f'(x)$ (f prime of x) to mean the derivative of f with respect to x. Other books and mathematicians sometimes use different notation for the derivative, such as

$$\frac{d}{dx}f(x) = y' = \frac{dy}{dx} = D_x(f(x)).$$

All of these different types of notation have essentially the same meaning: the derivative of a function with respect to x. Finding this derivative is called differentiating the function with respect to x.

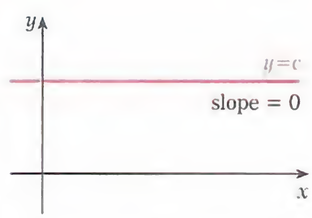
In stating the following rules, we assume that the functions f and g are differentiable.

Our first rule states that the derivative of a constant function is equal to zero.

THE DERIVATIVE OF A CONSTANT FUNCTION

If c is any real number, then $c' = 0$.

We can see this by considering the graph of the constant function $f(x) = c$, which is a horizontal line. The tangent line to a straight line at any point on the line coincides with the straight line itself. So, the slope of the tangent line is zero, and therefore the derivative is zero.



The slope of the tangent to the graph of $f(x) = c$, where c is constant, is zero.

We can also use the definition of the derivative to prove this result:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Example 14

a. If $f(x) = 13$, then $f'(x) = (13)' = 0$.

b. If $f(x) = -\frac{1}{2}$, then $f'(x) = \left(-\frac{1}{2}\right)' = 0$.

Next we consider how to find the derivative of any power function $f(x) = x^n$. Note that the rule applies not only to functions like $f(x) = x^3$, but also to those such as $g(x) = \sqrt[4]{x^3}$ and $h(x) = \dfrac{1}{x^5} = x^{-5}$.

THE DERIVATIVE OF A POWER FUNCTION (POWER RULE)
If n is any real number, then $(x^n)' = nx^{n-1}$.

Example 15

a. If $f(x) = x$, then $f'(x) = x' = 1 \cdot x^{1-1} = 1$.

b. If $f(x) = x^2$, then $f'(x) = (x^2)' = 2 \cdot x^{2-1} = 2x$.

c. If $f(x) = x^3$, then $f'(x) = (x^3)' = 3 \cdot x^{3-1} = 3x^2$.

Note

To differentiate a function containing a radical expression, we first convert the radical expression into exponential form, and then differentiate the exponential form using the Power Rule.

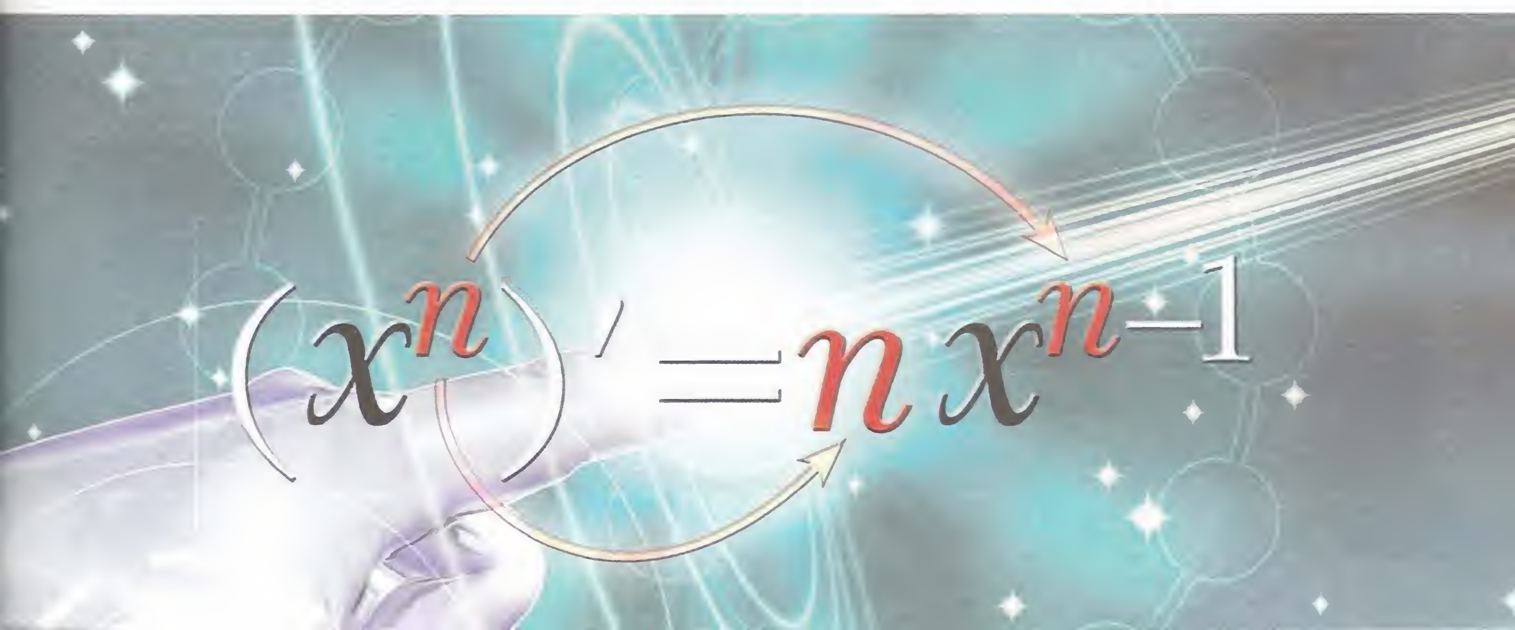
Example 16

a. If $f(x) = \sqrt[2]{x^3}$, then $f(x) = x^{3/2}$ in exponential form

$$f'(x) = (x^{3/2})' = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}.$$

b. If $f(x) = \dfrac{1}{x}$, then $f(x) = x^{-1}$ in exponential form

$$f'(x) = (x^{-1})' = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$



The proof of the Power Rule for the general case ($n \in \mathbb{R}$) is not easy to prove and will no be given here. However, we can prove the Power Rule for the case where n is a positive integer.

Proof (Power Rule) If $f(x) = x^n$, then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$.

Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$f'(x) = \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n \cdot (n-1)}{2} x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{n \cdot x^{n-1}h + \frac{n \cdot (n-1)}{2} x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$ (every term includes h as a factor, so h's can be simplified)

$$f'(x) = \lim_{h \to 0} \left[nx^{n-1} + \frac{n \cdot (n-1)}{2} x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] = n \cdot x^{n-1}$$ (if $h = 0$, then every term including h as a factor will be zero)

Check Yourself 5

Differentiate each function by using either the Constant Rule or the Power Rule.

1. $f(x) = 2$
2. $f(x) = 0.5$
3. $f(x) = -\frac{1}{3}$
4. $f(x) = \frac{\sqrt{3}}{2}$
5. $f(x) = x^3$
6. $f(x) = \sqrt[3]{x^7}$
7. $f(x) = \frac{1}{x^2}$
8. $f(x) = \frac{1}{\sqrt{x^3}}$

Answers

1. 0 2. 0 3. 0 4. 0 5. $3x^2$ 6. $\frac{7}{3}\sqrt[3]{x^4}$ 7. $-\frac{2}{x^3}$ 8. $-\frac{3}{2\sqrt{x^5}}$

The next rule states that the derivative of a constant multiplied by a differentiable function is equal to the constant times the derivative of the function.

THE CONSTANT MULTIPLE RULE

$$[c \cdot f(x)]' = c \cdot f'(x) \quad , \quad c \in \mathbb{R}$$

Example 17

a. If $f(x) = 3x$, then $f'(x) = (3x)' = 3 \cdot (x)' = 3 \cdot 1 = 3$.

b. If $f(x) = 3x^4$, then $f'(x) = (3x^4)' = 3(x^4)' = 3 \cdot (4x^3) = 12x^3$.

Proof (Constant Multiple Rule) If $g(x) = c \cdot f(x)$, then

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{c\cdot f(x+h)-c\cdot f(x)}{h}$$

$$g'(x) = c\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$g'(x) = c\cdot f'(x).$$

Example 18

a. If $f(x) = -\dfrac{2}{x^3}$, then $f'(x) = (-2x^{-3})' = -2(x^{-3})' = -2(-3x^{-1}) = 6x^{-4} = \dfrac{6}{x^4}$.

b. If $f(x) = 5\sqrt{x}$, then $f'(x) = (5x^{1/2})' = 5(x^{1/2})' = 5\left(\dfrac{1}{2}x^{-1/2}\right) = \dfrac{5}{2}x^{-1/2} = \dfrac{5}{2\sqrt{x}}$.

Next we consider the derivative of the sum or the difference of two differentiable functions. The derivative of the sum or the difference of two functions is equal to the sum or the difference of their derivatives. Note that the difference is also the sum since it deals with addition of a negative expression.

THE SUM RULE

$$[f(x) \mp g(x)]' = f'(x) \mp g'(x)$$

Note

We can generalize this rule for the sum of any finite number of differentiable functions.

$$[f(x) \mp g(x) \mp h(x) \mp \ldots]' = f'(x) \mp g'(x) \mp h'(x) \mp \ldots$$

Now, let's verify the rule for a sum of two functions.

Proof (Sum Rule) If $S(x) = f(x) + g(x)$, then

$$S'(x) = \lim_{h\to 0}\frac{S(x+h)-S(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$S'(x) = \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$S'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$S'(x) = f'(x)+g'(x).$$

Example 19 a. If $f(x) = x^{-2}+7$, then $f'(x) = (x^{-2}+7)' = (x^{-2})' + (7)' = -2x^{-3}+0 = -2x^{-3}$.

b. If $g(t) = \frac{t^2}{5} + \frac{5}{t^2}$, then $g'(t) = \left(\frac{t^2}{5}+5t^{-2}\right)' = \left(\frac{t^2}{5}\right)' + (5t^{-2})' = \frac{1}{5}(t^2)' + 5(t^{-2})'$

$$g(t)' = \frac{1}{5}(2t^{2-1}) + 5(-2t^{-2-1})$$

$$g(t)' = \frac{2}{5}t - 10t^{-3} = \frac{2t}{5} - \frac{10}{t^3}.$$

Notice that in this example, the independent variable is t instead of x. So, we differentiate the function $g(t)$ with respect to t.

By combining the Power Rule, the Constant Multiple Rule and the Sum Rule we can differentiate any polynomial. Let us look at some examples.

Example 20 Differentiate the polynomial function $f(x) = 3x^5 + 4x^4 - 7x^2 + 3x + 6$.

Solution

$$f'(x) = (3x^5 + 4x^4 - 7x^2 + 3x + 6)'$$

$$f'(x) = (3x^5)' + (4x^4)' + (-7x^2)' + (3x)' + (6)'$$

$$f'(x) = 3(x^5)' + 4(x^4)' - 7(x^2)' + 3(x)' + (6)'$$

$$f'(x) = 3\cdot 5x^4 + 4\cdot 4x^3 - 7\cdot 2x + 3\cdot 1 + 0$$

$$f'(x) = 15x^4 + 16x^3 - 14x + 3$$

Example 21 It is estimated that x months from now, the population of a certain community will be $P(x) = x^2 + 20x + 8000$.

a. At what rate will the population be changing with respect to time fifteen months from now?

b. How much will the population actually change during the sixteenth month?

Solution a. The rate of change of the population with respect to time is the derivative of the population function, i.e.

rate of change $= P'(x) = 2x + 20$.

Fifteen months from now the rate of change of the population will be:

$P'(15) = 2 \cdot 15 + 20 = 50$ people per month.

b. The actual change in the population during the sixteenth month is the difference between the population at the end of sixteen months and the population at the end of fifteen months. Therefore,

$$\begin{aligned}\text{the change in population} &= P(16) - P(15)\\ &= 8576 - 8525\\ &= 51 \text{ people.}\end{aligned}$$

Check Yourself 6

1. Find the derivative of each function with respect to the variable.

a. $f(x) = \dfrac{3}{2x}$ b. $f(r) = \dfrac{4}{3}\pi r^3$ c. $f(x) = 0.2\sqrt{x}$

d. $f(x) = 3x^2 + 5x - 1$ e. $f(t) = \dfrac{4}{t^3} - \dfrac{t^2}{3} + t$ f. $f(x) = \dfrac{x^3 - 4x^2 + 3}{x}$

2. Find the derivative of $f(x) = \dfrac{x^3 - 3x^2 + 3x - 1}{x - 1}$.

3. Differentiate $f(x) = \dfrac{x^2\sqrt{x} - \sqrt{x}}{x\sqrt{x} + \sqrt{x}}$.

Answers

1. a. $-\dfrac{3}{2x^2}$ b. $4\pi r^2$ c. $\dfrac{0.1}{\sqrt{x}}$ d. $6x + 5$ e. $-\dfrac{12}{t^4} - \dfrac{2t}{3} + 1$ f. $2x - 4 - \dfrac{3}{x^2}$

2. $2x - 2$ 3. 1

B. THE PRODUCT AND THE QUOTIENT RULES

Now we learn how to differentiate a function formed by multiplication or division of functions. Based on your experience with the Constant Multiple and Sum Rules we learned in the preceding part, you may think that the derivative of the product of functions is the product of separate derivatives, but this guess is wrong. The correct formula was discovered by Leibniz and is called the Product Rule.

The Product Rule states that the derivative of the product of two functions is the derivative of the first function times the second function plus the first function times the derivative of the second function.

THE PRODUCT RULE

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Be careful! The derivative of the product of two functions is not equal to the product of the derivatives:

$[f(x)g(x)]' \neq f'(x)g'(x)$

We can easily see this by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then

$$f(x)g(x) = x \cdot x^2 = x^3 \qquad f'(x) = 1 \text{ and } g'(x) = 2x$$

$$[f(x)g(x)]' = 3x^2 \qquad f'(x)g'(x) = 1 \cdot 2x = 2x$$

$$[f(x)g(x)]' \neq f'(x)g'(x).$$

Example 22 Find the derivative of the function $f(x) = x(x+1)$.

Solution By the Product Rule,

$$f'(x) = x \cdot (x+1)' + (x)' \cdot (x+1) = x \cdot 1 + 1 \cdot (x+1) = 2x + 1.$$

We can check this result by using direct computation:

$f(x) = x(x+1) = x^2 + x$ so, $f'(x) = 2x + 1$, which is the same result.

Note that preferring direct differentiation when it is easy to expand the brackets is always simpler than applying the Product Rule.

Example 23 Differentiate the function $f(x) = (2x^2+1)(x^2-x)$.

Solution

$$f'(x) = (2x^2+1)' \cdot (x^2-x) + (2x^2+1) \cdot (x^2-x)'$$

$$f'(x) = (4x)(x^2-1) + (2x^2+1)(2x-1)$$

$$f'(x) = 4x^3 - 4x + 4x^3 - 2x^2 + 2x - 1$$

$$f'(x) = 8x^3 - 2x^2 - 2x - 1$$

Example 24 Differentiate the function $f(x) = (x^3 + x - 2)(2\sqrt{x} + 1)$.

Solution First, we convert the radical part into exponential form:

$$f(x) = (x^3 + x - 2)(2\sqrt{x} + 1) = (x^3 + x - 2) \cdot (2x^{1/2} + 1).$$

Now, by the Product Rule,

$$f'(x) = (x^3 + x - 2)' \cdot (2x^{1/2} + 1) + (x^3 + x - 2) \cdot (2x^{1/2} + 1)'$$

$$f'(x) = (3x^2 + 1)(2x^{1/2} + 1) + (x^3 + x - 2) \cdot x^{-1/2} = 6x^{5/2} + 3x^2 + 2x^{1/2} + 1 + x^{5/2} + x^{1/2} - 2x^{-1/2}$$

$$f'(x) = 7x^{5/2} + 3x^2 + 3x^{1/2} - 2x^{-1/2} + 1.$$

Let us look at the proof of the Product Rule.

Proof (Product Rule) If $P(x) = f(x)g(x)$, then

$$P'(x) = \lim_{h\to 0} \frac{P(x+h) - P(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

By adding $-f(x+h)g(x) + f(x+h)g(x)$ (which is zero) to the numerator and factoring, we have:

$$P'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$P'(x) = \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$$P'(x) = \lim_{h\to 0} \left(f(x+h)\left[\frac{g(x+h) - g(x)}{h}\right] + g(x)\left[\frac{f(x+h) - f(x)}{h}\right]\right)$$

$$P'(x) = \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$P'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x) = f'(x)g(x) + f(x)g'(x).$$

Example 25 Differentiate the function $f(x) = (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)$.

Solution In this example we have a product of three functions, but we are only able to apply the rule for the product of two functions. So, before we proceed we must imagine the function as a product of two functions as follows:

$$f(x) = (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)$$

$$f'(x) = \underbrace{[(x^2 + 1)(3x^4 - 5x)]'}_{\text{requires product rule once more}}(x^3 + 2x^2 + 4) + (x^2 + 1)(3x^4 - 5x)(x^3 + 2x^2 + 4)'$$

$$f'(x) = [2x(3x^4 - 5x) + (x^2 + 1)(12x^3 - 5)](x^3 + 2x^2 + 4) + (x^2 + 1)(3x^4 - 5x)(3x^2 + 4x).$$

Our aim is to introduce this method and because any further simplification is time consuming, we will stop at this point.

The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. Or,

$$\left(\frac{\text{numerator}}{\text{denominator}}\right)' = \frac{\text{derivative of the numerator} \times \text{denominator} - \text{numerator} \times \text{derivative of the denominator}}{\text{the square of the denominator}}.$$

THE QUOTIENT RULE

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \qquad g(x) \neq 0$$

$$\left(\frac{f(x)}{g(x)}\right)' \neq \frac{f'(x)}{g'(x)}$$

The quotient rule is probably the most complicated formula you will have to learn in this text. It may help if you remember that the quotient rule resembles the Product Rule.

Also note that like in the Product Rule, the derivative of a quotient is not equal to the quotient of derivatives.

Example 26 Find the derivative of the function $f(x) = \dfrac{3x+1}{2x-1}$.

Solution Using the Quotient Rule:

$$f'(x) = \frac{(3x+1)'(2x-1) - (3x+1)(2x-1)'}{(2x-1)^2}$$

$$f'(x) = \frac{3 \cdot (2x-1) - (3x+1) \cdot 2}{(2x-1)^2} = \frac{6x - 3 - 6x - 2}{(2x-1)^2}$$

$$f'(x) = -\frac{5}{(2x-1)^2}.$$

Example 27 Differentiate the rational function $f(x) = \dfrac{x^2 + x - 21}{x-1}$.

Solution According to the Quotient Rule,

$$f'(x) = \frac{(2x+1) \cdot (x-1) - (x^2 + x - 21) \cdot 1}{(x-1)^2}$$

$$f'(x) = \frac{2x^2 - x - 1 - x^2 - x + 21}{(x-1)^2} = \frac{x^2 - 2x + 20}{(x-1)^2}$$

$$f'(x) = \frac{x^2 - 2x + 20}{x^2 - 2x + 1}.$$

Example 28 Differentiate the function $f(x)=\dfrac{2x^2+3x+1}{2x}$.

Solution Before trying to use the Quotient Rule let us simplify the formula of the function:

$$f(x)=\frac{2x^2+3x+1}{2x}=\frac{2x^2}{2x}+\frac{3x}{2x}+\frac{1}{2x}=x+\frac{3}{2}+\frac{1}{2}x^{-1}.$$

In this example, finding the derivative will be easier and quicker without using the Quotient Rule.

$$f'(x)=1+0-\frac{1}{2}x^{-2}=1-\frac{1}{x^2}=\frac{x^2-1}{x^2}$$

Note

We do not need to use the Quotient Rule every time we differentiate a quotient. Sometimes performing division gives us an expression which is easier to differentiate than the quotient.

Let us verify the Quotient Rule.

Proof (Quotient Rule) Let $Q(x)=\dfrac{f(x)}{g(x)}$ and $Q(x)$ be differentiable.

We can write $f(x) = Q(x)g(x)$.

If we apply the Product Rule: $f'(x) = Q'(x)g(x) + Q(x)g'(x)$

Solving this equation for $Q'(x)$, we get

$$Q'(x)=\frac{f'(x)-Q(x)g'(x)}{g(x)}=\frac{f'(x)-\dfrac{f(x)}{g(x)}\cdot g'(x)}{g(x)}$$

$$Q'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}.$$

Example 29 $f(x) = \sqrt{x}\cdot g(x)$, where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

Solution $f'(x)=(\sqrt{x}\cdot g(x))'=(\sqrt{x})'\cdot g(x)+\sqrt{x}\cdot g'(x)$

$$f'(x)=\frac{g(x)}{2\sqrt{x}}+\sqrt{x}\cdot g'(x)$$

So $f'(4)=\dfrac{g(4)}{2\sqrt{4}}+\sqrt{4}\cdot g'(4)=\dfrac{2}{2\cdot 2}+2\cdot 3=\dfrac{13}{2}.$

Check Yourself 7

1. Find the derivative of each function using the Product or the Quotient Rule.

a. $f(x) = 2x(x^2 + x + 1)$

b. $f(x) = (x^3 - 1)(x^2 - 2)$

c. $f(x) = \left(\frac{1}{x^2} + x\right)\left(\frac{1}{x} + 1\right)$

d. $f(x) = (\sqrt{x} + 1)\left(x^2 + \frac{1}{\sqrt{x}}\right)$

e. $f(x) = \frac{2x+4}{3x-1}$

f. $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$

g. $f(x) = \frac{x^2 - x + 10}{x+1}$

h. $f(x) = \frac{x^3 + 3x^2 - 5x + 6}{2x}$

2. If $f(x)$ is a differentiable function, find an expression for the derivative of each function.

a. $y = x^2 f(x)$

b. $y = \frac{f(x)}{x^2}$

c. $y = \frac{x^2}{f(x)}$

d. $y = \frac{1 + xf(x)}{x}$

3. Suppose that f and g are two functions such that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$ and $g'(5) = 2$. Find each value.

a. $(fg)'(5)$

b. $\left(\frac{f}{g}\right)'(5)$

c. $\left(\frac{g}{f}\right)'(5)$

Answers

1. a. $6x^2 + 4x + 2$ b. $5x^4 - 6x^2 - 2x$ c. $1 - \frac{2}{x^3} - \frac{3}{x^4}$ d. $\frac{5}{2}x\sqrt{x} + 2x - \frac{1}{2x\sqrt{x}}$

e. $-\frac{14}{(3x-1)^2}$ f. $\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$ g. $\frac{x^2 + 2x - 11}{(x+1)^2}$ h. $x - \frac{3}{x^2} + \frac{3}{2}$

2. a. $2xf(x) + x^2 f'(x)$ b. $\frac{f'(x) \cdot x^2 - 2xf(x)}{x^4}$ c. $\frac{2xf(x) - f'(x) \cdot x^2}{(f(x))^2}$ d. $\frac{x^2 f'(x) - 1}{x^2}$

3. a. -16 b. $-\frac{20}{9}$ c. 20

C. THE CHAIN RULE

We have learned how to find the derivatives of expressions that involve the sum, difference, product or quotient of different powers of x. Now consider the function given below.

$$h(x) = (x^2 + x - 1)^{50}$$

In order to differentiate $h(x)$ using the rules we know, we need to expand $h(x)$, then find the derivative of each separate term. This method is, however, tedious!

Consider also the function $m(x) = \sqrt{x^2 + x - 1}$. This function is also difficult to differentiate using the rules we have learned. For each of the two functions $h(x)$ and $m(x)$, the differentiation formula we learned in the previous sections cannot be applied easily to calculate the derivatives $h'(x)$ and $m'(x)$.

We know that both h and m are composite functions because both are built up from simpler functions. For example, the function $h(x) = (x^2 + x - 1)^{50}$ is built up from the two simpler functions $f(x) = x^{50}$ and $g(x) = x^2 + x - 1$ like this:

$$h(x) = f(g(x)) = [g(x)]^{50} = (x^2 + x - 1)^{50}$$

Here we know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $h = f(g(x))$ in terms of the derivatives of f and g.

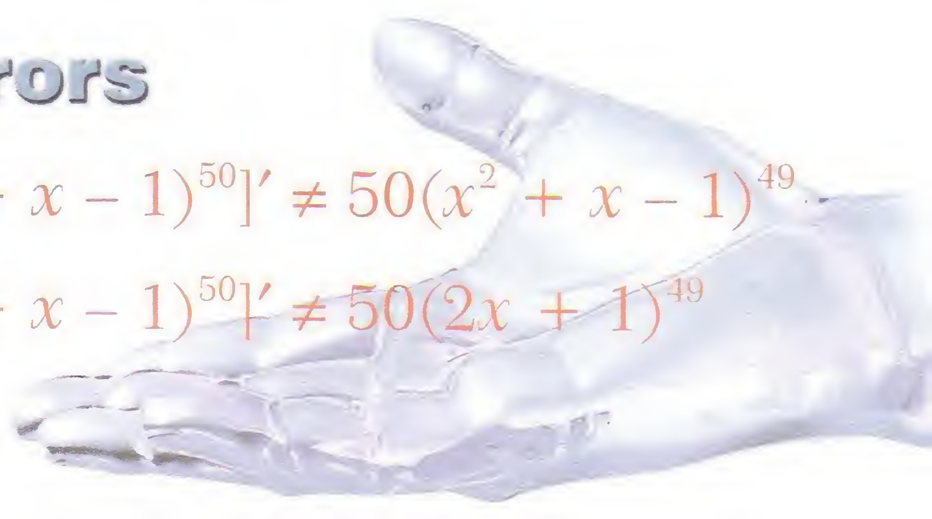
THE CHAIN RULE

$$[\underbrace{f}_{\substack{\text{outer}\\ \text{function}}}(\underbrace{g(x)}_{\substack{\text{argument}\\ \text{of the}\\ \text{outer}\\ \text{function}}})]' = \underbrace{f'(g(x))}_{\substack{\text{derivative of}\\ \text{the outer}\\ \text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative of}\\ \text{the argument}\\ \text{of the outer}\\ \text{function}}}$$

For example, if $h(x) = f(g(x)) = (x^2 + x - 1)^{50}$, then
$h'(x) = f'(g(x)) \cdot g'(x) = 50(x^2 + x - 1)^{49} \cdot (2x + 1)$.

Common Errors

1. $[(x^2 + x - 1)^{50}]' \neq 50(x^2 + x - 1)^{49}$
2. $[(x^2 + x - 1)^{50}]' \neq 50(2x + 1)^{49}$



Example 30 Differentiate the function $f(x) = (3x^2 + 5x)^{2005}$.

Solution By the Chain Rule, $f'(x) = 2005(3x^2+5x)^{2004} \cdot (3x^2+5x)' = 2005(3x^2+5x)^{2004} \cdot (6x+5)$.

Example 31 Suppose $m(x) = f(g(x))$ and $g(1) = 5$, $g'(1) = 2$, $f(5) = 3$ and $f'(5) = 4$ are given. Find $m'(1)$.

Solution By the Chain Rule, $m'(x) = f'(g(x)) \cdot g'(x)$. So $m'(1) = f'(g(1)) \cdot g'(1) = f'(5) \cdot 2 = 4 \cdot 2 = 8$.

Note

The Chain Rule can be generalized for the composition of more than two functions as follows:

$$[f_1(f_2(f_3(\ldots f_n(x)\ldots)))]' = [f_1'(f_2(f_3(\ldots f_n(x)\ldots)))] \cdot (f_2'(f_3(\ldots f_n(x)\ldots))) \cdot (f_3'(\ldots f_n(x)\ldots)) \cdot \ldots \cdot f_n'(x)$$

Using the Chain Rule we can generalize the Power Rule as follows:

GENERAL POWER RULE

$$[(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$$

By using this rule we can more easily differentiate the functions that can be written as the power of any other functions.

Example 32 Differentiate the function $m(x) = \sqrt{x^2 + x - 1}$.

Solution We can rewrite the function as $m(x) = (x^2 + x - 1)^{\frac{1}{2}}$ and apply the General Power Rule:

$$m'(x) = \frac{1}{2}(x^2 + x - 1)^{-\frac{1}{2}} \cdot (x^2 + x - 1)'$$

$$m'(x) = \frac{1}{2}(x^2 + x - 1)^{-\frac{1}{2}} \cdot (2x + 1)$$

$$m'(x) = \frac{2x+1}{2\sqrt{x^2 + x - 1}}$$

Example 33 Differentiate the function $f(x) = \dfrac{1}{x^2 + 3x}$.

Solution

$$f'(x) = [(x^2 + 3x)^{-1}]' = -1(x^2 + 3x)^{-2} \cdot (x^2 + 3x)'$$

$$f'(x) = -1(x^2 + 3x)^{-2} \cdot (2x + 3)$$

$$f'(x) = -\frac{2x+3}{(x^2 + 3x)^2}$$

Example 34 Differentiate the function $f(x) = (2x^3 + x^2 - 15)^{-\frac{1}{3}}$.

Solution

$$f'(x) = -\frac{1}{3}(2x^3 + x^2 - 15)^{-\frac{4}{3}} \cdot (2x^3 + x^2 - 15)'$$

$$f'(x) = -\frac{1}{3} \cdot \frac{1}{\sqrt[3]{(2x^3 + x^2 - 15)^4}} \cdot (6x^2 + 2x)$$

$$f'(x) = -\frac{6x^2 + 2x}{3\sqrt[3]{(2x^3 + x^2 - 15)^4}}$$

Example 35 Differentiate the function $f(x) = ((x+1)^{-\frac{2}{3}} + 5x)^{-3}$.

Solution

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot ((x+1)^{-\frac{2}{3}} + 5x)'$$

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot (-\frac{2}{3}(x+1)^{-\frac{5}{3}} \cdot (x+1)' + 5)$$

$$f'(x) = -3((x+1)^{-\frac{2}{3}} + 5x)^{-4} \cdot (5 - \frac{2}{3}(x+1)^{-\frac{5}{3}})$$

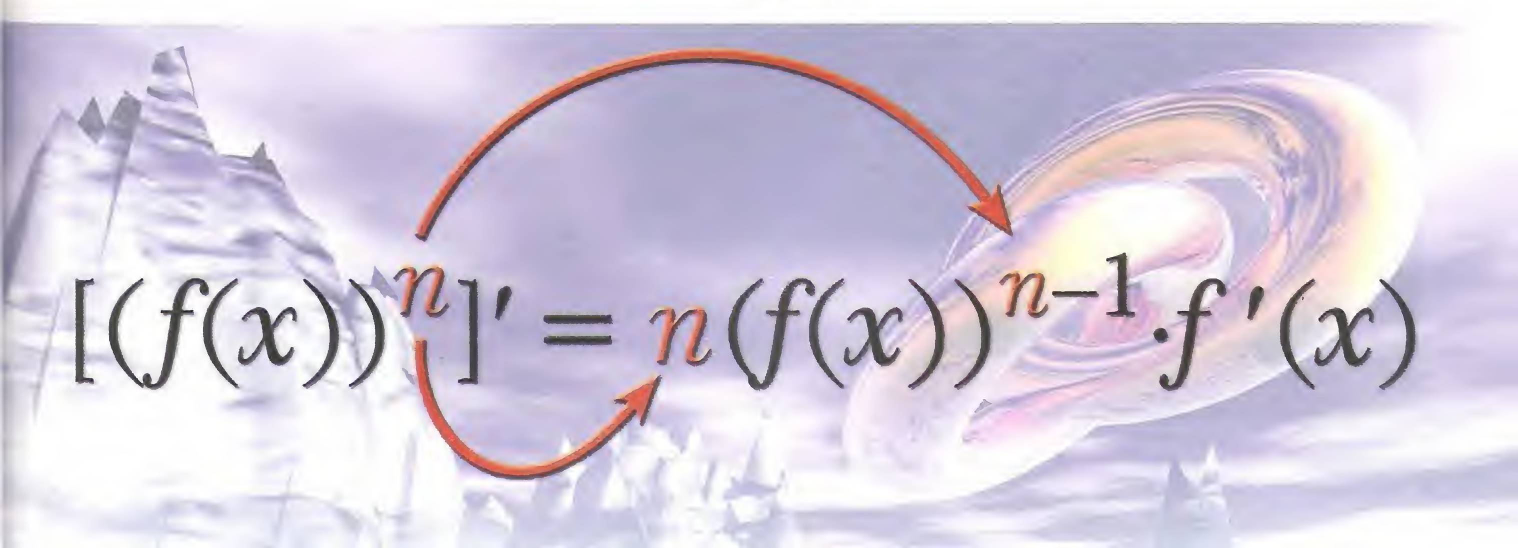
Example 36 Differentiate the function $f(x) = (2x-3)^5 \cdot \sqrt{x^2 - 2x}$.

Solution The function is the product of two expressions, so we can use the Product Rule:

$$f'(x) = ((2x-3)^5)' \cdot \sqrt{x^2 - 2x} + (2x-3)^5 \cdot (\sqrt{x^2 - 2x})'$$

$$f'(x) = 5 \cdot (2x-3)^4 \cdot 2 \cdot \sqrt{x^2 - 2x} + (2x-3)^5 \cdot \frac{1}{2} \cdot (x^2 - 2x)^{-\frac{1}{2}} \cdot (2x-2)$$

$$f'(x) = 10(2x-3)^4\sqrt{x^2 - 2x} + \frac{(2x-3)^5 \cdot (2x-2)}{2\sqrt{x^2 - 2x}}$$



Example 37 Differentiate the function $g(t)=\left(\dfrac{2t+1}{t-3}\right)^7$.

Solution

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\left(\frac{2t+1}{t-3}\right)' \qquad \textit{(by the Power Rule)}$$

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\frac{2\cdot(t-3)-1\cdot(2t+1)}{(t-3)^2} \qquad \textit{(by the Quotient Rule)}$$

$$g'(t)=7\left(\frac{2t+1}{t-3}\right)^6\cdot\frac{2t-6-2t-1}{(t-3)^2} \qquad \textit{(simplify)}$$

$$g'(t)=\frac{-49(2t+1)^6}{(t-3)^8}.$$

Notation

Remember that if $y=f(x)$, then we can denote its derivative by y' or $\dfrac{dy}{dx}$.

If $y=f(g(x))$ such that $y=f(u)$ and $u=g(x)$, then we can denote the derivative of $f(g(x))$ by $y'=f'(g(x))\cdot g'(x)$ or $y'=f'(u)\cdot u'(x)$ or $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$.

The notation $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ is called Leibniz notation for the Chain Rule.

Example 38 Given that $y=u^2-1$ and $u=3x^2+1$, find $\dfrac{dy}{dx}$ by using the Chain Rule.

Solution By the Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{d}{du}(u^2-1)\cdot\frac{d}{dx}(3x+1)$$ *(find the derivative of the first function with respect to u and the second function with respect to x)*

$$\frac{dy}{dx}=(2u-1)\cdot 3$$

$$\frac{dy}{dx}=(2\cdot(3x+1)-1)\cdot 3$$

$$\frac{dy}{dx}=18x+3.$$

Check Yourself 8

1. Find the derivative of $f(x) = (2x + 1)^3$.
2. Differentiate $y = (x^3 - 1)^{100}$.
3. Find $f'(x)$ given $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$.
4. Find the derivative of $g(x) = \sqrt[4]{\dfrac{x^3-1}{x^3+1}}$.
5. $y = \dfrac{1}{u}$ and $u = 3x - 1$ are given. Find $\dfrac{dy}{dx}$.

Answers

1. $6(2x + 1)^2$
2. $300x^2(x^3 - 1)^{99}$
3. $\dfrac{2x+1}{3\sqrt[3]{(x^2+x+1)^4}}$
4. $\dfrac{1}{4}(\dfrac{x^3+1}{x^3-1})^{\frac{3}{4}}\dfrac{6x^2}{(x^3+1)^2}$
5. $-\dfrac{3}{(3x-1)^2}$

D. HIGHER ORDER DERIVATIVES

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Look at three different ways of writing the second derivative of a function:

$$f''(x) = y'' = \frac{d^2y}{dx^2}$$

Example 39 Find the second derivative of the function $f(x) = \dfrac{2x}{x-1}$.

Solution By the Quotient Rule,

$$f'(x) = \frac{2 \cdot (x-1) - 2x \cdot 1}{(x-1)^2} = \frac{2x-2-2x}{(x-1)^2} = -\frac{2}{(x-1)^2}.$$

Now differentiate $f'(x)$ to get $f''(x)$:

$$f''(x) = \left(-\frac{2}{(x-1)^2}\right)' = (-2(x-1)^{-2})' = 4(x-1)^{-3} \cdot 1 = \frac{4}{(x-1)^3}$$

Note

Before computing the second derivative of a function, always try to simplify the first derivative as much as possible. Otherwise the computation of the second derivative will be more tedious.

Notation

If we differentiate the second derivative $f''(x)$ of a function $f(x)$ one more time, we get the third derivative $f'''(x)$. Differentiate again and we get the fourth derivative, which we write as $f^{(4)}(x)$ since the prime notation $f''''(x)$ begins to get difficult to read. In general, the derivative obtained from $f(x)$ after n successive differentiations is called the nth derivative or the derivative of order n and written by $f^{(n)}(x)$ or $\dfrac{d^ny}{dx^n}$.

Example 40 Find the derivatives of all orders of the polynomial function

$f(x) = x^5 + 4x^4 + 2x^3 - 5x^2 - 6x + 7$.

Solution

$f'(x) = 5x^4 + 16x^3 + 6x^2 - 10x - 6$

$f''(x) = 20x^3 + 48x^2 + 12x - 10$

$f'''(x) = 60x^2 + 96x + 12$

$f^{(4)}(x) = 120x + 96$

$f^{(5)}(x) = 120$

$f^{(n)}(x) = 0 \quad (\text{for } n > 5)$

Example 41 Find a general expression for the nth derivative of the function $f(x)=\dfrac{1}{x}$.

Solution

$$\frac{dy}{dx}=(x^{-1})'=-x^{-2}=-\frac{1}{x^2}$$

$$\frac{d^2y}{dx^2}=(-x^{-2})'=2x^{-3}=\frac{2}{x^3}$$

$$\frac{d^3y}{dx^3}=(2x^{-3})'=-6x^{-4}=-\frac{6}{x^4}$$

$$\frac{d^4y}{dx^4}=(-6x^{-4})'=24x^{-5}=\frac{24}{x^5}$$

$$\frac{d^5y}{dx^5}=(24x^{-5})'=-120x^{-6}=-\frac{120}{x^6}$$

$$\frac{d^ny}{dx^n}=(\ldots)'=(-1)^n\cdot n!\,x^{-(n+1)}=\frac{(-1)^n n!}{x^{n+1}}$$

> $n!=n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$ for any natural number n

Check Yourself 9

1. Find the second derivative of each function.

   a. $f(x) = x^3 - 3x^2 + 4x + 5$ b. $f(x)=\dfrac{x-1}{x+2}$

2. Find the third derivative of each function.

   a. $f(x) = x^{2/3}$ b. $f(t)=(\frac{1}{2}t^2-1)^5$

Answers

1\. a. $6x - 6$ b. $-\dfrac{6}{(x+2)^3}$ 2. a. $\dfrac{8}{27}x^{-\frac{7}{3}}$ b. $\dfrac{15t(t^2-2)^2(3t^2-2)}{2}$

LINEAR MOTION

Motion is one of the key subjects in physics. We define many concepts and quantities to explain the motion in one dimension. We use some formulas to state the relations between the quantities. Derivative plays an important role in defining the quantities and producing the formulas from other derivatives. Here we will give the definition of important concepts and formulas that includes the uses of derivative.

DISPLACEMENT

The displacement is the change in the position of an object. If we denote the position at time t_1 by x_1, and the position at time t_2 by x_2, then the displacement is the difference between these two points; this is defined by

$$\Delta x = x_2 - x_1.$$

The time interval is, similarly,

$$\Delta t = t_2 - t_1.$$

We use the capital Greek letter Δ(delta) to show a change in a variable from one value to another.

VELOCITY

The velocity describes how fast the position of an object changes. It is measured over a certain time interval. If a car has a displacement Δx in a particular time interval Δt, then the car's average velocity, V_{av}, over that time interval is defined by

$$V_{av} = \frac{\textit{displacement}}{\textit{time}\text{ interval}} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}.$$

The definition of the average velocity includes a time interval. We learn more about the motion when smaller time intervals are used. Because of this, we define the instantaneous velocity as follows.

The instantaneous velocity at a time t is the velocity of an object at that given instant of time. In other words, it is the limit of the average velocity as Δt approaches zero:

$$V(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

From the definition given above, we may conclude that the instantaneous velocity is the derivative of the displacement with respect to time t.

$$V(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

ACCELERATION

The term acceleration refers to the rate of change in velocity of an object with respect to time. We define the average acceleration, a_{av}, in terms of velocity v_1 at time t_1 and v_2 at time t_2:

$$a_{av} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

Now we will define the instantaneous acceleration as follows.

The instantaneous acceleration is the limit of the expression $\frac{\Delta v}{\Delta t}$ as the time interval goes to zero.

$$a(t) \lim_{\Delta t \to 0} = \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

This means that the instantaneous acceleration is the derivative of velocity with respect to time. Also it is the second derivative of the displacement.

For example, the position function $x(t)$ of a car moving along a straight line is given as $x(t) = 4t^2 + 6t - 20$ m where t is in seconds.

The derivative of the position function gives the velocity function.

$$V(t) = \frac{dx}{dt} = (4t^2 + 6t - 20)' = 8t + 6 \text{ m/s}$$

The acceleration is

$$a(t) = \frac{dv}{dt} = (8t + 6)' = 8 \text{ m/s}^2.$$

So, the car moves with constant acceleration.

EXERCISES 1.2

A. Basic Differentiation Rules

1. Find the derivative of each function by using the rules of differentiation.

a. $f(x) = \sqrt{2}$

b. $f(x) = -\dfrac{1}{151}$

c. $f(x) = e^{\pi}$

d. $f(x) = \dfrac{1}{12}x^8$

e. $f(x) = 2x^{0.8}$

f. $f(x) = \dfrac{5}{4}x^{4/5}$

g. $f(x) = \dfrac{2}{\sqrt[6]{4^{11}}}$

h. $f(x) = 0.3x^{0.7}$

i. $f(x) = 7x^{-12}$

j. $f(x) = 5x^2 - 3x + 7$

k. $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x}$

l. $f(x) = \dfrac{4}{t^4} - \dfrac{3}{t^3} + \dfrac{2}{t}$

m. $f(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}$

n. $f(x) = \sqrt{x} + \dfrac{2}{\sqrt{x}} + \dfrac{1}{x}$

o. $f(x) = 1 - \dfrac{1}{x} + \dfrac{3}{\sqrt{x}}$

B. The Product and The Quotient Rules

2. Find the derivative of each function by using the Product or the Quotient Rule.

a. $f(x) = 5x(x^2 - 1)$

b. $f(x) = (2x + 3)(3x - 4)$

c. $f(x) = 10(3x + 1)(1 - 5x)$

d. $f(x) = (x^3 - 1)(x + 1)$

e. $f(x) = (x^3 - x^2 + x - 1)(x^2 + 2)$

f. $f(x) = (1 + \sqrt{t})(2t^2 - 3)$

g. $f(x) = \dfrac{3}{2x + 4}$

h. $f(x) = \dfrac{x - 1}{2x + 1}$

i. $f(x) = \dfrac{1 - 2x}{1 + 3x}$

j. $f(x) = \dfrac{\sqrt{x}}{x^2 + 1}$

k. $f(x) = \dfrac{x^2 + 2}{x^2 + x + 1}$

l. $f(x) = \dfrac{x + \sqrt{3x}}{3x - 1}$

3. Given that $f(1) = 2$, $f'(1) = -1$, $g(1) = -2$ and $g'(1) = 3$, find the value of $h'(1)$.

a. $h(x) = f(x) \cdot g(x)$

b. $h(x) = (x^2 + 1) \cdot g(x)$

c. $h(x) = \dfrac{xf(x)}{x + g(x)}$

d. $h(x) = \dfrac{f(x) \cdot g(x)}{f(x) - g(x)}$

4. Differentiate the function $f(x)=\dfrac{x-3x\sqrt{x}}{\sqrt{x}}$ by simplifying and by the Quotient Rule. Show that both of your answers are equivalent. Which method do you prefer? Why?

5. $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$ and $g'(3) = 5$ are given. Find the value of the following expressions.

a. $(f + g)'(3)$ b. $(fg)'(3)$

c. $\left(\dfrac{f}{g}\right)'(3)$ d. $\left(\dfrac{f}{f-g}\right)(3)$

C. The Chain Rule

6. Find the derivative of each function.

a. $f(x) = (3x - 1)^2$

b. $f(x) = (x^2 + 2)^5$

c. $f(x) = (x^5 - 3x^2 + 6)^7$

d. $f(x) = (x - 2)^{-3}$

e. $f(x)=\dfrac{2}{(5x^2+3x-1)^2}$

f. $f(x)=\dfrac{1}{\sqrt{4x^2+1}}$

g. $f(x)=(\sqrt{x+1}+\sqrt{x})^3$

h. $f(x) = (x - 1)^5 \cdot (3x + 1)^{1/3}$

i. $f(x)=\dfrac{(1-3x)^7}{(2x+1)^4}$

j. $f(x)=(\dfrac{3x-9}{2x+4})^3$

k. $f(x)=\sqrt{\dfrac{2x-1}{3x+1}}$

l. $f(x) = 3x + [2x^2 + (x^3 + 1)^2]^{3/4}$

7. $h(x) = g(f(x))$ and $f(2) = 3$, $f'(2) = -3$, $g(3) = 5$ and $g'(3) = 4$ are given. Find $h'(2)$.

8. By using the Chain Rule, find $\dfrac{dy}{dx}$ for each function.

a. $y = u^2 - 1$, $u = 2x + 1$

b. $y = u^2 + 2u + 2$, $u = x - 1$

c. $y=\dfrac{1}{u-1}$, $u = x^3$

d. $y=\sqrt{u}+\dfrac{1}{\sqrt{u}}$, $u = x^2 - x$

D. Higher Order Derivatives

9. Find the second derivative of each function.

a. $f(x) = 3x^2 - 7x + 2$

b. $f(x) = (x^2 + 1)^7$

c. $f(x)=\dfrac{x^2}{x-1}$

d. $f(x)=\sqrt{2x-1}$

10. Find the third derivative of each function.

a. $f(x) = 5x^4 - 3x^3$

b. $f(x) = \dfrac{2}{x}$

c. $f(x)=\sqrt{3x-2}$

d. $f(x) = (2x - 3)^4$

Mixed Problems

11. Find the equation of the tangent line to the graph of the function $f(x) = (x^3 + 1)(3x^2 - 4x + 2)$ at the point $(1, 2)$.

12. The curve $y = \dfrac{x}{x^2+1}$ is called a serpentine curve. Find the equation of the tangent line to the curve at the point $x = 3$.

13. f is a differentiable function. Find an expression for the derivative of each of the following functions.

a. $y = x^2\sqrt{x}f(x)$

b. $y = x^3(f(x))^2$

c. $y = \dfrac{x^3}{f(x)}$

d. $y = \dfrac{x + xf(x)}{\sqrt{x}}$

14. Prove that $(fgh)' = f'gh + fg'h + fgh'$ if f, g and h are differentiable functions.

15. A scientist adds a toxin to a colony of bacteria. He estimates that the population of the colony after t hours will be $P(t) = \dfrac{24t+10}{t^2+1}$ thousand bacteria. Find the estimated rate of change of the population after three hours.

16. The concentration of a certain drug in a patient's bloodstream t hours after injection is given by $C(t) = \dfrac{0.2t}{t^2+1}$.

a. Find the rate at which the concentration of the drug is changing with respect to time.

b. How fast is the concentration changing 1 hour after the injection? What about after 2 hours?

17. $g(x) = f(x^2 + 1)$ is given. Find $g'(1)$ if $f'(2) = 3$.

18. Find an expression for the derivative of $f\left(\dfrac{g(x)h(x)}{m(n(x))}\right)$ if f, g, h, m and n are differentiable functions.

19. If the tangent to the graph of f at point $(2, 3)$ has an angle of $60°$ with x-axis, find the slope of tangent to the graph of $g(x) = f^2(x) - x \cdot f(x)$ at $x = 2$.

20. Given that $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$ and $f''(a) = -\dfrac{7}{64}$, find a.

21. Given that $f(4 \cdot g(x) + 7) = x^3 - 2x^2 + 3$ and $g(x) = 1 - x$, find $f'(-1)$.

22. Find an expression for the nth derivative of the function $f(x) = \dfrac{1}{2x}$.

3 DERIVATIVES OF ELEMENTARY FUNCTIONS

A. DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this section, we will study the derivative formulas for exponential and logarithmic function. Let us begin with the derivative formulas for exponential functions.

1. Exponential Functions

DERIVATIVE OF NATURAL EXPONENTIAL FUNCTION

$$(e^x)' = e^x$$

Proof (Derivative of Natural Exponential Function)

Let $f(x) = e^x$, then by the definition of derivatives

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h}.$$

We are unable to evaluate this limit using the techniques we have learned before. But the calculations in the following table helps us to guess it correctly.

$\rightarrow h \leftarrow$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{e^h-1}{h}$	0.9516	0.9950	0.9995	1	1.0005	1.0050	1.0517

The table tells us that, $\lim\limits_{h\to 0}\frac{e^h-1}{h}=1.$

So, $f'(x) = e^x \cdot 1 = e^x$.

Example 42 Find the derivative of the function $f(x) = x^2e^x$.

Solution By the Product Rule,

$$f'(x) = (x^2e^x)' = (x^2)'e^x + x^2(e^x)' = 2xe^x + x^2e^x = xe^x(x+2).$$

Example 43 Find the derivative of the function $f(x) = (e^x + 1)^{\frac{5}{2}}$.

Solution By the General Power Rule,

$$f'(x) = \frac{5}{2}(e^x + 1)^{\frac{3}{2}} \cdot (e^x + 1)' = \frac{5}{2}e^x(e^x + 1)^{\frac{3}{2}}.$$

CHAIN RULE FOR NATURAL EXPONENTIAL FUNCTION

$$(e^{f(x)})' = e^{f(x)} \cdot f'(x)$$

Example 44 Find the derivative of the function $f(x) = e^{3x+1}$.

Solution By the Chain Rule,

$$f'(x) = e^{3x+1} \cdot (3x+1)' = 3e^{3x+1}.$$

Example 45 Find $\dfrac{dy}{dx}$ if $f(x) = xe^{x^2-1}$.

Solution By the Product Rule,

$$\frac{dy}{dx} = 1 \cdot e^{x^2-1} + x \cdot e^{x^2-1} \cdot (x^2 - 1)' = e^{x^2-1} + 2x^2e^{x^2-1} = e^{x^2-1}(2x^2 + 1).$$

DERIVATIVE OF EXPONENTIAL FUNCTION

$$(a^x)' = a^x \ln a \quad , \quad a \in \mathbb{R}^+$$

Proof (Derivative of Exponential Function)

By using the identity $a = e^{\ln a}$ we can rewrite the expression as

$f(x) = a^x = e^{\ln a^x} = e^{x \ln a}$ *(by the property of logarithm)*

$f'(x) = e^{x\ln a} \cdot \ln a = e^{\ln a^x} \cdot \ln a$

$f'(x) = a^x \ln a.$

Example 46 Given that $f(x) = 3^x$, find $f'(x)$.

Solution $f'(x) = 3^x \cdot \ln 3$

Example 47 Find the derivative of the function $f(x) = e^x \cdot 2^x$.

Solution By the Product Rule,

$f'(x) = (e^x)' \cdot 2^x + e^x \cdot (2^x)' = e^x 2^x + e^x 2^x \ln 2 = (2e)^x(\ln 2 + 1)$.

CHAIN RULE FOR EXPONENTIAL FUNCTION
$(a^{f(x)})' = a^{f(x)} \cdot \ln a \cdot f'(x) \quad , \qquad a \in \mathbb{R}^+$

Example 48 Find the derivative of the function $f(x) = 5^{x^2+1}$.

Solution By the Chain Rule,

$f'(x) = 5^{x^2+1} \cdot \ln 5 \cdot (x^2+1)' = (\ln 5) \cdot 2x \cdot 5^{x^2+1}$.

Example 49 If $f(x) = \dfrac{3^{2x+1}}{x^2+1}$, find $f'(x)$.

Solution

$$f'(x) = \frac{(3^{2x+1})' \cdot (x^2+1) - 3^{2x+1} \cdot (x^2+1)'}{(x^2+1)^2} \quad \textit{(by the Quotient Rule)}$$

$$f'(x) = \frac{3^{2x+1} \cdot \ln 3 \cdot 2 \cdot (x^2+1) - 2x \cdot 3^{2x+1}}{(x^2+1)^2} \quad \textit{(by the Chain Rule)}$$

$$f'(x) = \frac{2 \cdot 3^{2x+1}[(x^2+1) \cdot \ln 3 - x]}{(x^2+1)^2}$$

Check Yourself 10

Find the derivative of each function.

1. $f(x) = 2xe^x$
2. $g(x) = e^{x^3+3x^2+1}$
3. $f(x) = e^{x^3-e^x}$
4. $f(x) = e^{2x-1}$
5. $g(x) = (x+1)e^{x^2-1}$
6. $f(x) = e^x \cdot 3^x$
7. $g(x) = 3^{x^2+x+1}$
8. $h(x) = \dfrac{3^{4x+1}}{x^2-1}$

Answers

1. $2e^x(x+1)$
2. $e^{x^3+3x^2+1} \cdot (3x^2+6x)$
3. $e^{x^3-e^x} \cdot (3x^2 - e^x)$
4. $2e^{2x-1}$
5. $e^{x^2-1} \cdot (2x^2+2x+1)$
6. $e^x \cdot 3^x(1+\ln 3)$
7. $3^{x^2+x+1} \cdot \ln 3 \cdot (2x+1)$
8. $\dfrac{2 \cdot 3^{4x+1} \cdot (2\ln 3 \cdot (x^2-1) - x)}{(x^2-1)^2}$

2. Logarithmic Functions

DERIVATIVE OF NATURAL LOGARITHMIC FUNCTION

$$(\ln x)' = \frac{1}{x} \quad , \quad x > 0$$

Example 50 Find the derivative of the function $f(x) = x \ln x$.

Solution By the Product Rule,

$$f'(x) = (x)' \cdot \ln x + x \cdot (\ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Example 51 Find the derivative of the function $f(x) = \ln \sqrt[5]{x^3}$.

Solution We can write $\sqrt[5]{x^3} = x^{\frac{3}{5}}$. Now,

$$f'(x) = (\ln\ x^{\frac{3}{5}})' = (\frac{3}{5}\ln\ x)' = \frac{3}{5} \cdot \frac{1}{x} = \frac{3}{5x}.$$

Example 52 Find the derivative of the function $f(x) = (\ln x + x)^2$.

Solution Applying the General Power Rule,

$f'(x) = 2(\ln\ x + x) \cdot (\ln\ x + x)'$ *(by the General Power Rule)*

$f'(x) = 2(\ln\ x + x)\left(\frac{1}{x} + 1\right)$ *(by the Sum Rule and the derivative of ln (x))*

CHAIN RULE FOR NATURAL LOGARITHMIC FUNCTION

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

Example 53 Find the derivative of the function $f(x) = \ln(x^2 + 3x + 1)$.

Solution $f'(x) = \dfrac{(x^2 + 3x + 1)'}{x^2 + 3x + 1} = \dfrac{2x + 3}{x^2 + 3x + 1}.$

DERIVATIVE OF LOGARITHMIC FUNCTION

$$(\log_a x)' = \frac{1}{x \ln a} \quad , \quad x > 0, \quad a > 0, \quad a \neq 1$$

Proof (Derivative of Logarithmic Function)

Let $f(x) = \log_a x$, then $a^{f(x)} = x$.

Differentiating both sides of the equation $a^{f(x)} = x$ with respect to x, we get

$a^{f(x)}(\ln a) f'(x) = 1$.

Note that we are looking for $f(x)$.

$$f'(x) = \frac{1}{a^{f(x)} \ln a} = \frac{1}{x \ln a}$$

Example 54 Find the derivative of the function $f(x) = \log_2 x$.

Solution $f'(x) = \dfrac{1}{x \ln 2}$

CHAIN RULE FOR LOGARITHMIC FUNCTION

$$(\log_a f(x))' = \frac{f'(x)}{f(x) \cdot \ln a}$$

Example 55 Find the derivative of the function $f(x) = \log_2(x^3 + 2x)$.

Solution $f'(x) = \dfrac{(x^3 + 2x)'}{(x^3 + 2x) \cdot \ln 2} = \dfrac{3x^2 + 2}{(x^3 + 2x) \ln 2}$

Example 56 Find the derivative of the function $f(x) = e^{x^2+3x} \cdot \log_3(2x - 4)$.

Solution By the Product Rule,

$f'(x) = (e^{x^2+3x})' \cdot \log_3(2x - 4) + e^{x^2+3x} \cdot (\log_3(2x - 4))'$

$f'(x) = e^{x^2+3x} \cdot (2x + 3) \cdot \log_3(2x - 4) + e^{x^2+3x} \cdot \dfrac{2}{(2x - 4)\ln 3}$.

Check Yourself 11

Find the derivative of each function.

1. $f(x) = x^2 \ln x$
2. $g(x) = \ln \sqrt[7]{x^5}$
3. $h(x) = (e^x + \ln x)^2$
4. $f(x) = \ln(x^2 - 5x + 1)$
5. $h(x) = [\ln(x^2 + x + 1)]^2$
6. $f(x) = \ln(e^x + 2)$
7. $f(x) = \log_2(x^2 + 5x - 1)$
8. $g(x) = \log_3[\ln(x^2 + 1)]$
9. $g(x) = \log^2(e^x + x - 1)$

Answers

1. $x(2\ln x + 1)$
2. $\dfrac{5}{7x}$
3. $2(e^x + \ln x)\,(e^x + \dfrac{1}{x})$
4. $\dfrac{2x-5}{x^2-5x+1}$
5. $2\ln(x^2 + x + 1) \cdot (\dfrac{2x+1}{x^2+x+1})$
6. $\dfrac{e^x}{e^x+2}$
7. $\dfrac{2x+5}{(x^2+5x-1)\ln 2}$
8. $\dfrac{2x}{(x^2+1)[\ln(x^2+1)]\ln 3}$
9. $\dfrac{2(e^x+1)\log(e^x+x-1)}{(e^x+x-1)\ln 10}$

3. Logarithmic Differentiation

Sometimes the task of finding the derivative of a complicated function involving products, quotients, or powers can be made easier by first applying the laws of logarithms to simplify it. This technique is called **logarithmic differentiation**. Let us look at some examples.

Example 57 $f(x) = x(3x - 1)(x^2 + 3)$ is given. Find the first derivative of f by using logarithmic differentiation.

$\ln(ab) = \ln a + \ln b$
$a, b > 0$

Solution $f(x) = x(3x - 1)(x^2 + 3)$

$\ln f(x) = \ln [x(3x - 1)(x^2 + 3)]$ *(take the logarithms is of both sides)*

$\ln f(x) = \ln(x) + \ln(3x - 1) + \ln(x^2 + 3)$ *(write logarithms as sums)*

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{3}{3x-1} + \frac{2x}{x^2+3}$$ *(take derivatives of both sides)*

$$f'(x) = f(x)\left(\frac{1}{x} + \frac{3}{3x-1} + \frac{2x}{x^2+3}\right)$$ *(isolate $f'(x)$)*

$$f'(x) = x(3x-1)(x^2+3)\left(\frac{1}{x} + \frac{3}{3x-1} + \frac{2x}{x^2+3}\right)$$ *(substitute for $f(x)$)*

Example 58 Find the derivative of the function $f(x)=\dfrac{x^{\frac{3}{5}}\cdot\sqrt[3]{x^2+1}}{(2x+1)^7}$.

Solution First find the logarithms of both sides of the expression:

$$\ln f(x)=\ln\left(\frac{x^{\frac{3}{5}}\cdot\sqrt[3]{x^2+1}}{(2x+1)^7}\right)=\frac{3}{5}\ln x+\frac{1}{3}\ln(x^2+1)-7\ln(2x+1).$$

$\ln(\frac{a}{b})=\ln a-\ln b$

$\ln a^p=p\ln a$

$a,b>0$

Now differentiate both sides of the equation with respect to x:

$$\frac{f'(x)}{f(x)}=\frac{3}{5}\cdot\frac{1}{x}+\frac{1}{3}\cdot\frac{1}{x^2+1}\cdot 2x-\frac{7\cdot 2}{(2x+1)}$$

$$f'(x)=f(x)\left(\frac{3}{5x}+\frac{2x}{3x^2+3}-\frac{14}{2x+1}\right)=\left(\frac{x^{\frac{3}{5}}\cdot\sqrt[3]{x^2+1}}{(2x+1)^7}\right)\left(\frac{3}{5x}+\frac{2x}{3x^2+3}-\frac{14}{2x+1}\right)$$

If we had not used logarithmic differentiation here, finding the derivative would have been a long and complicated process.

Example 59 Given that $x>0$, find the derivative of $f(x)=x^x$.

Solution First find the logarithms of both sides of the expression:

$\ln f(x)=\ln x^x=x\cdot\ln x.$

Differentiate both sides of the equation with respect to x:

$$\frac{f'(x)}{f(x)}=x'\cdot\ln x+x\cdot(\ln x)'=1+\ln x$$

$$f'(x)=f(x)(1+\ln x)=x^x(1+\ln x).$$

Check Yourself 12

Find the derivative of each function by using logarithmic differentiation.

1. $f(x)=(3x-1)^5(x^3+1)^6$
2. $f(x)=\dfrac{e^{x^2+1}\cdot(x^2-1)^{10}}{\sqrt{x}}$
3. $f(x)=\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$
4. $g(x)=x^{\frac{1}{x}}$
5. $h(x)=x^{\ln x}$
6. $f(x)=(\ln x)^{\frac{1}{\ln x}}$

Answers

1. $\left(\dfrac{15}{3x-1}+\dfrac{18x^2}{x^3+1}\right)(3x-1)^5(x^3+1)^6$
2. $\left(2x+\dfrac{20x}{x^2-1}-\dfrac{1}{2x}\right)\cdot\dfrac{e^{x^2+1}\cdot(x^2-1)^{10}}{\sqrt{x}}$
3. $\left(-\dfrac{x}{(x^2-1)(x^2+1)}\right)\cdot\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$
4. $\dfrac{1-\ln x}{x^2}\cdot x^{\frac{1}{x}}$
5. $\dfrac{2\ln x}{x}\cdot x^{\ln x}$
6. $\dfrac{1-\ln(\ln x)}{x\ln^2 x}\cdot(\ln x)^{\frac{1}{\ln x}}$

B. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Let us begin by looking at the derivatives of the sine and cosine functions.

DERIVATIVES OF SINE AND COSINE

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

Example 60 Find the derivative of the function $f(x) = (\sin x + \cos x)^2$.

Solution

$f'(x) = 2(\sin x + \cos x)(\sin x + \cos x)'$ *(by the General Power Rule)*

$f'(x) = 2(\sin x + \cos x)(\cos x - \sin x)$ *(by the sum, derivative of the sine and cosine)*

$f'(x) = 2(\cos^2 x - \sin^2 x)$ *(simplify)*

$f'(x) = 2\cos 2x$ *(by the trigonometric identity)*

$\cos 2x = \cos^2 x - \sin^2 x$

Now let us derive the formula for the derivative of the function $f(x) = \sin x$.

Proof (Derivative of Sine Function)

By the definition of the derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} = \lim_{h\to 0}\left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right) = \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h} + \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$f'(x) = \sin x\cdot 0 + \cos x\cdot 1 = \cos x.$$

$\lim_{h\to 0}\frac{\sin h}{h} = 1$

$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Example 61 Find the derivative of the function $f(x) = x\cdot\sin x$.

Solution By the Product Rule

$$f'(x) = (x\cdot\sin x)' = (x)'\cdot\sin x + x\cdot(\sin x)' = \sin x + x\cos x.$$

Example 62 Find the derivative of the function $f(x)=\dfrac{e^x}{\cos x}$.

Solution $f'(x)=\left(\dfrac{e^x}{\cos x}\right)'=\dfrac{(e^x)'\cdot\cos x-e^x\cdot(\cos x)'}{\cos^2 x}$ *(by the Quotient Rule)*

$f'(x)=\dfrac{e^x\cos x+e^x\sin x}{\cos^2 x}$ *(differentiate)*

$f'(x)=\dfrac{e^x(\cos x+\sin x)}{\cos^2 x}$ *(simplify)*

CHAIN RULE FOR SINE AND COSINE

$$(\sin f(x))' = \cos f(x)\cdot f'(x)$$
$$(\cos f(x))' = -\sin f(x)\cdot f'(x)$$

Example 63 Find the derivative of the function $f(x) = \cos(x^3 - x)$.

Solution $f'(x)=(\cos(x^3-x))'=-\sin(x^3-x)\cdot(x^3-x)'=-\sin(x^3-x)\cdot(3x^2-1)$

Example 64 Find the derivative of the function $f(x) = \sin^3 x^2$.

Solution In this example we have the composition of three functions.

$f(x)=\sin^5 x^2=(\sin(x^2))^5$

We apply the Chain Rule beginning from the outermost function:

$f'(x)=((\sin(x^2))^5)'=5(\sin(x^2))^4\cdot(\sin(x^2))'$

$f'(x)=5(\sin(x^2))^4\cdot\cos(x^2)\cdot(x^2)'$

$f'(x)=5(\sin(x^2))^4\cdot\cos(x^2)\cdot 2x$

$f'(x)=10x\sin(x^2)\cos(x^2)$

$\sin 2x = 2\sin x\cos x$

Example 65 Find the derivative of the function $f(x) = (\sin(e^x) - \cos(\ln x))^{100}$.

Solution $f'(x) = [(\sin(e^x) - \cos(\ln x))^{100}]'$

$f'(x) = 100\cdot(\sin(e^x)-\cos(\ln x))^{99}\cdot(\sin(e^x)-\cos(\ln x))'$ *(by the General Power Rule)*

$f'(x) = 100\cdot(\sin(e^x)-\cos(\ln x))^{99}\cdot\left(e^x\cos(e^x)+\dfrac{\sin(\ln x)}{x}\right)$ *(by the Chain Rule for sine and cosine)*

Check Yourself 13

Find the derivative of each function.

1. $f(x) = x - 3\sin x$
2. $f(x) = x\cos x$
3. $f(x) = \dfrac{\sin x}{1+\cos x}$
4. $f(x) = \dfrac{e^x}{\sin x + \cos x}$
5. $f(x) = \cos^2(x^2 + x - 1)$
6. $f(x) = \sin(e^x + x^2)$
7. $f(x) = \cos^2(\ln x + 1)$
8. $f(x) = e^x \sin(e^x)$

Answers

1. $1 - 3\cos x$
2. $\cos x - x\sin x$
3. $\dfrac{1}{1+\cos x}$
4. $\dfrac{2 \cdot e^x \cdot \sin x}{(\sin x + \cos x)^2}$
5. $-\sin(2x^2 + 2x - 2) \cdot (2x + 1)$
6. $\cos(e^x + x^2) \cdot (e^x + 2x)$
7. $\dfrac{-\sin(2\ln x + 2)}{x}$
8. $e^x(\sin(e^x) + e^x\cos(e^x))$

DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS	
$(\tan x)' = \sec^2 x = 1 + \tan^2 x$	$(\tan f(x))' = \sec^2 f(x) \cdot f'(x)$
$(\cot x)' = -\csc^2 x = -(1 + \cot^2 x)$	$(\cot f(x))' = -\csc^2 f(x) \cdot f'(x)$
$(\sec x)' = \sec x \cdot \tan x = \dfrac{\sin x}{\cos^2 x}$	$(\sec f(x))' = \sec f(x) \cdot \tan f(x) \cdot f'(x)$
$(\csc x)' = -\csc x \cdot \cot x = -\dfrac{\cos x}{\sin^2 x}$	$(\csc f(x))' = -\csc f(x) \cdot \cot f(x) \cdot f'(x)$

Example 66 Find the derivative of the function $f(x) = \tan(x^2 - 3x + 1)$.

Solution $f'(x) = \sec^2(x^2 - 3x + 1) \cdot (x^2 - 3x + 1)' = \sec^2(x^2 - 3x + 1) \cdot (2x - 3)$

or $= (1 + \tan^2(x^2 - 3x + 1))(2x - 3)$.

Example 67 Find the derivative of the function $f(x) = \csc(\ln x)$.

Solution $f'(x) = -\csc(\ln x) \cdot \cot(\ln x) \cdot (\ln x)' = -\csc(\ln x) \cdot \cot(\ln x) \cdot \dfrac{1}{x} = -\dfrac{\csc(\ln x) \cdot \cot(\ln x)}{x}$

Example 68 Find the derivative of the function $f(x) = e^{-\cot^2 x}$.

Solution $f'(x) = e^{-\cot^2 x} \cdot (-\cot^2 x)' = e^{-\cot^2 x} \cdot (-2\cot x)' \cdot (\cot x)' = e^{-\cot^2 x} \cdot (-2\cot x) \cdot (-\csc^2 x)$

$f'(x) = 2e^{-\cot^2 x} \cdot \cot x \cdot \csc^2 x$

Example 69 Find the derivative of the function $f(x)=\dfrac{\sec x}{1+\tan x}$.

Solution By the Quotient Rule,

$$f'(x)=\frac{(\sec x)'\cdot(1+\tan x)-\sec x\cdot(1+\tan x)'}{(1+\tan x)^2} \quad \textit{(by the Quotient Rule)}$$

$$f'(x)=\frac{\sec x\tan x\cdot(1+\tan x)-\sec x\cdot\sec^2 x}{(1+\tan x)^2} \quad \textit{(differentiate)}$$

$$f'(x)=\frac{\sec x(\tan x+\tan^2 x-\sec^2 x)}{(1+\tan x)^2} \quad \textit{(factorize)}$$

$$f'(x)=\frac{\sec x(\tan x-1)}{(1+\tan x)^2} \quad \textit{(simplify using } \tan^2 x+1=\sec^2 x\textit{)}$$

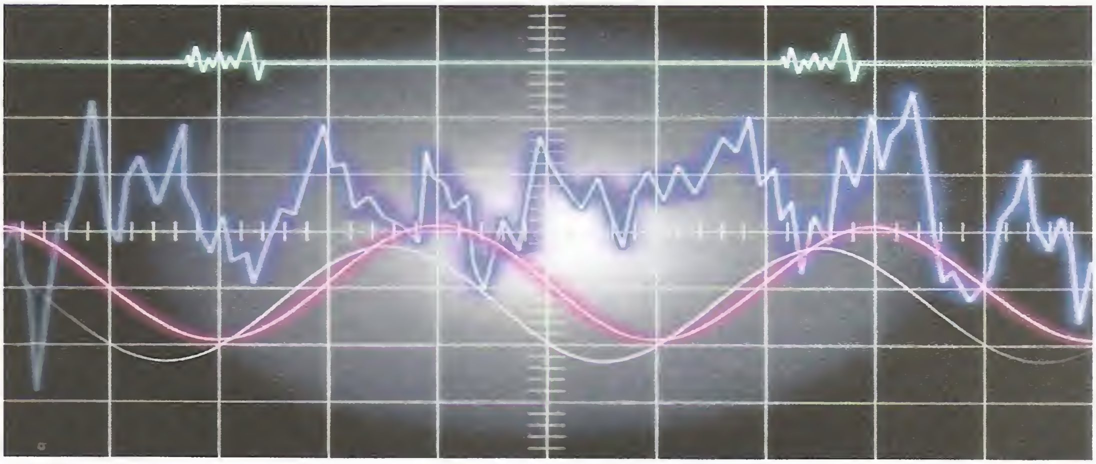
Check Yourself 14

Find the derivative of each function.

1. $f(x)=\dfrac{\tan x}{x}$
2. $f(x) = 4\sec x - \cot x$
3. $f(x) = e^x\csc x$
4. $f(x) = \ln(\tan x)$
5. $f(x) = \tan^2(\ln x)$
6. $f(x) = \cot(x^2 - x + 1)$

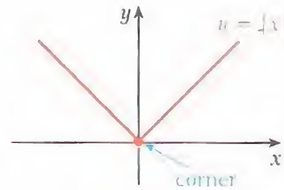
Answers

1. $\dfrac{x\sec^2 x-\tan x}{x^2}$
2. $4\sec x\tan x + \csc^2 x$
3. $e^x\csc x(1 - \cot x)$
4. $\dfrac{1}{\sin x\cos x}$
5. $\dfrac{2\tan(\ln x)\cdot\sec^2(\ln x)}{x}$
6. $\csc^2(x^2 - x + 1)\cdot(1 - 2x)$



C. DERIVATIVES OF SPECIAL FUNCTIONS

1. Absolute Value Functions



If $f(x) = |g(x)|$, then $f'(x) = \begin{cases} g'(x), & g(x) > 0 \\ -g'(x), & g(x) < 0 \end{cases}$.

In general, a derivative does not exist when the function has 0 as value because of a 'corner' as demonstrated in the graph on the right.

When $g(x) = 0$, the derivative exists only when the right-hand side and the left-hand side derivatives are equal to each other.

We can also formulate the derivative expression as follows:

$$f'(x) = \frac{|g(x)|}{g(x)} \cdot g'(x) = \operatorname{sgn}[g(x)] \cdot g'(x), \quad g(x) \neq 0.$$

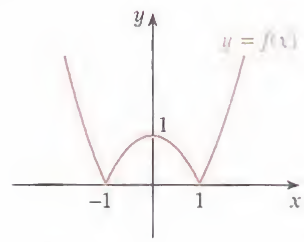
Example 70 Find the derivative of the function $f(x) = |1 - x^2|$ at the points $x = 2$, $x = 1$ and $x = 0$.

Solution Let us begin by trying to rewrite the function as a piecewise function.

The roots of the function are $x = -1$ and $x = 2$.

Then the function f will be $f(x) = \begin{cases} x^2 - 1, & x < -1 \text{ and } x \geq 1 \\ 1 - x^2, & -1 \leq x < 1 \end{cases}$.

A piecewise function is a function that is defined by different formulae in different parts of its domain.



Let us find $f'(2)$:

Note that $f(2) \neq 0$.

For $x = 2$, $f'(x) = (x^2 - 1)' = 2x$.

So, $f'(2) = 2 \cdot 2 = 4$.

Let us find $f'(1)$:

Since $f(1) = 0$, we will check the left-hand and the right-hand derivatives.

For $x < 1$, $f'(x) = (1 - x^2)' = -2x$.

So, $f'(1^-) = -2 \cdot (1) = -2$.

For $x > 1$, $f'(x) = (1 - x^2)' = 2x$.

So, $f'(1^+) = 2 \cdot 1 = 2$.

Since $f'(1^-) \neq f'(1^+)$, $f'(1)$ does not exist.

Let us find $f'(0)$:

Note that $f(0) \neq 0$.

For $x = 0$, $f'(x) = (1 - x^2)' = -2x$.

So, $f'(0) = -2 \cdot 0 = 0$.

Example 71 Given that $f(x) = |x^3 - 4x^2 + 4x|$, find the derivative of $f(x)$ at the point $x = 2$.

Solution The piecewise form of the function is $f(x) = \begin{cases} -x^3 + 4x^2 - 4x, x < 0 \\ x^3 - 4x^2 + 4x, \ x \geq 0 \end{cases}$.

Since $f(2) = 0$, we will check the left-hand and the right-hand derivatives.

For $0 \leq x < 2$, $f'(x) = (x^3 - 4x^2 + 4x)' = 3x^2 - 8x + 4$

$f'(2^-) = 3 \cdot (2)^2 - 8 \cdot 2 + 4 = 0.$

For $x > 2$, $f'(x) = (x^3 - 4x^2 + 4x)' = 3x^2 - 8x + 4$

$f'(2^+) = 3 \cdot (2)^2 - 8 \cdot 2 + 4 = 0.$

Since the left-hand and the right-hand derivatives are equal to each other, the derivative of the function exists at the point $x = 2$ and $f'(2) = 0$.

Example 72 Given that $f(x) = |x - x^2|$, find $f'(2)$ and $f'(3)$.

Solution Since $f(2) \neq 0$ and $f(3) \neq 0$, we can use the formula $f'(x) = \frac{|x - x^2|}{x - x^2} \cdot (1 - 2x)$.

$$f'(2) = \frac{|2 - 2^2|}{2 - 2^2} \cdot (1 - 2 \cdot 2) = -1 \cdot (-3) = 3$$

$$f'(3) = \frac{|3 - 3^2|}{3 - 3^2} \cdot (1 - 2 \cdot 3) = -1 \cdot (-5) = 5$$

Example 73 Given that $f(x) = |\cos x|$, find the derivative of $f(x)$ at the points $x = \frac{\pi}{3}$ and $x = \pi$.

Solution For $x = \frac{\pi}{3}$, $\cos x > 0$. So, $f(x) = \cos x$ and $f'(x) = -\sin x$.

$$f'(\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$

For $x = \pi$, $\cos x < 0$. So, $f(x) = -\cos x$ and $f'(x) = \sin x$.

$f'(\pi) = \sin \pi = 0$

Example 74 Given that $f(x) = |x^3 - 9| + x^2$, find $f''(2)$.

Solution For $x = 2$, $x^3 - 9 < 0$ and so $f(x) = -x^3 + 9 + x^2$.

If we take the derivative twice,

$f'(x) = -3x^2 + 2x$

$f''(x) = -6x + 2.$

Therefore, $f''(2) = -6 \cdot 2 + 2 = -10.$

Check Yourself 15

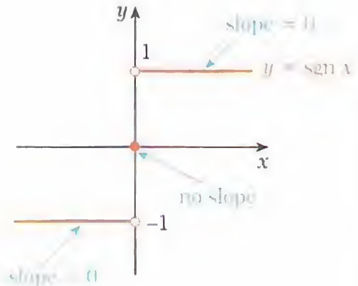
1. Given that $f(x) = |x^2 - 3x|$, find $f'(3)$ and $f'(5)$.
2. Given that $f(x) = |x^4 - 2x^2 + 1|$, find the derivative of $f(x)$ at the point $x = 1$.
3. Given that $f(x) = |\sin x|$, find $f'(\frac{\pi}{6})$ and $f'(\frac{\pi}{2})$.

Answers

1. does not exist; 7 2. 0 3. $\frac{\sqrt{3}}{2}$; 0

2. Sign Functions

Note that a sign function has the range $\{-1, 0, 1\}$. When it takes -1 and 1 as its value, the graph is a horizontal line. Since the slope of a horizontal line is 0, we have 0 as the derivative. When the function takes 0 as its value, the graph has a discontinuity. So, the derivative does not exist. Look at the graph on the right:



In conclusion, if $f(x) = \text{sgn}\,(g(x))$, then

$$f'(x) = \begin{cases} 0, & g(x) \neq 0 \\ \text{does not exist,} & g(x) = 0 \end{cases}.$$

Example 75 Given that $f(x) = \text{sgn}(x^2 - x)$, find the derivative of $f(x)$ at the points $x = -2$ and $x = 1$.

Solution We begin by finding the value of $f(-2)$ and $f(1)$:

$f(-2) = \text{sgn}((-2)^2 - (-2)) = \text{sgn}(6) = 1$

$f(1) = \text{sgn}(1^2 - 1) = \text{sgn}(0) = 0.$

Since $f(-2) \neq 0$, $f'(-2) = 0$.

Since $f(1) = 0$, $f'(1)$ does not exist ($f(x)$ is not continuous at $x = 1$).

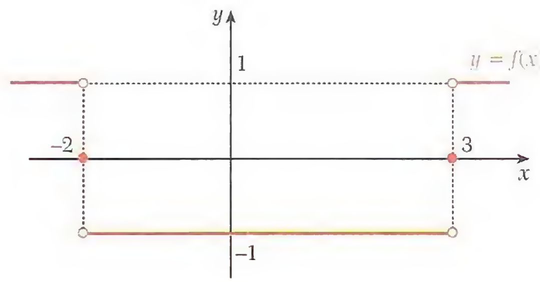
Example 76 Find the largest interval on which the function $f(x) = \text{sgn}(x^2 - x - 6)$ is differentiable.

Solution We can rewrite the function as a piecewise function:

$$f(x) = \begin{cases} 1, & x < -2 \text{ and } x > 3 \\ -1, & -2 < x < 3 \\ 0, & x = -2 \text{ and } x = 3 \end{cases}.$$

Since $f(x)$ is not continuous at the points $x = -2$ and $x = 3$, it is not differentiable. It has a derivative at all other points, and this is equal to zero.

So, the largest interval on which f is differentiable is $\mathbb{R} \setminus \{-2, 3\}$.



3. Floor Functions

If $f(x) = [\![g(x)]\!]$, then $f'(x) = \begin{cases} 0, & g(x) \notin \mathbb{Z} \\ \text{may not exist}, & g(x) \in \mathbb{Z} \end{cases}$.

When $g(x) \in \mathbb{Z}$, $f(x)$ is certainly continuous and differentiable. However, when $g(x) \notin \mathbb{Z}$, we cannot be certain. It may be differentiable or not. In order to determine whether a floor function is differentiable or not at a given value, we check the left-hand and the right-hand derivatives.

Example 77 Given that $f(x) = [\![\frac{2x+1}{3}]\!]$, find the derivative of $f(x)$ at the points $x = 2$ and $x = 4$.

Solution For $x = 2$, $\frac{2x+1}{3} = \frac{5}{3} \notin \mathbb{Z}$. Since the expression $\frac{2x+1}{3}$ is not an integer for $x = 2$, $f'(2) = 0$.

For $x = 4$, $\frac{2x+1}{3} = 3 \in \mathbb{Z}$. Here we have to find the left-hand and the right-hand derivatives, because the expression $\frac{2x+1}{3}$ is an integer for $x = 4$.

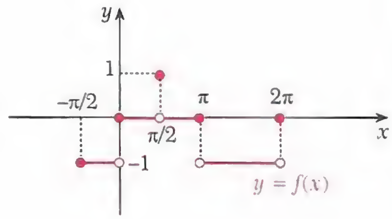
$$f'(4^-) = \lim_{x \to 4^-} \frac{f(x) - f(4)}{x - 4} = \frac{2-3}{4^- - 4} = \frac{-1}{0^-} = +\infty$$

$$f'(4^+) = \lim_{x \to 4^-} \frac{f(x) - f(4)}{x - 4} = \frac{3-3}{4^+ - 4} = \frac{0}{0^+} = 0$$

Since $f'(4^-) \neq f'(4^+)$, $f'(4)$ does not exist.

Example 78 Given that $f(x) = [\![\sin x]\!]$, find $f'(\frac{\pi}{6})$ and $f'(\pi)$.

Solution $\sin(\frac{\pi}{6}) = \frac{1}{2} \notin \mathbb{Z}$. So, $f'(\frac{\pi}{6}) = 0$.



At $x = \pi$, $f(x)$ is not continuous (see the graph). So, it is not differentiable. Thus the derivative of $f(x)$ does not exist at this point.

Example 79 Given that $f(x) = [\![x^2]\!]$, find $f'(0)$.

Solution For $x = 0$, $f(x) = 0 \in \mathbb{Z}$. So, we have to find the left-hand and the right-hand derivatives of the function $f(x) = [\![x^2]\!]$ at the point $x = 0$.

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{[\![x^2]\!] - 0}{x - 0} = \frac{0}{0^-} = 0$$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{[\![x^2]\!] - 0}{x - 4} = \frac{0}{0^+} = 0$$

Notice that if $-1 < x < 1$, $[\![x^2]\!] = 0$.

Since $f'(0^-) = f'(0^+)$, the derivative of $f(x) = [\![x^2]\!]$ exists at the point $x = 0$ and $f'(0) = 0$.

Example 80 Given that $f(x) = x|x| + [\![x]\!]\text{sgn}(x)$, find $f'(\frac{3}{2})$.

Solution For $x = \left(\frac{3}{2}\right)^-$, we can rewrite the function as $f(x) = x \cdot x + 1 \cdot 1 = x^2 + 1$.

For $x = \left(\frac{3}{2}\right)^-$, we have the same function $f(x) = x \cdot x + 1 \cdot 1 = x^2 + 1$.

So, for $x = \frac{3}{2}$ we have $f'(x) = 2x$. That gives $f'(\frac{3}{2}) = 2 \cdot \frac{3}{2} = 3$.

Check Yourself 16

1. Given that $f(x) = \text{sgn}\,(x^2 + x)$, find $f'(-1)$ and $f'(2)$.
2. Given that $f(x) = [\![\cos x]\!]$, find $f'(\frac{\pi}{3})$ and $f'(\frac{\pi}{2})$.
3. Given that $f(x) = |x^2 + 3x - 4| + [\![x^2]\!] + \text{sgn}\,(x^2 - 1)$, find $f'(0)$.

Answers

1. does not exist; 0 2. 0; does not exist 3. −3

D. IMPLICIT DIFFERENTIATION

Up to now we have worked with the functions expressed in the form $y = f(x)$. In this form, the variable y is expressed easily in terms of the variable x. A function in this form is said to be in the **explicit form**. However, some functions cannot be written in explicit form. Consider the following equation:

$$y^5 + y + x = 0$$

If we are given a value of x, we can calculate y in this equation. However, we cannot write the equation in the form $y = f(x)$. We say that x determines y **implicitly**, and that y is an **implicit function** of x. Look at the same more implicit functions:

$$x^5 + 2xy^2 - 3y^4 = 7$$
$$y - 2y^2 = x$$
$$x^2 - y^2 + 4y = 0$$

How can we differentiate an implicit function? Recall the Chain Rule for differentiation. In an implicit function, y is still a function of x, even if we cannot write this explicitly. So, we can use the Chain Rule to differentiate terms containing y as functions of x. For example, if we are differentiating in terms of x,

$$(y^4)' = [(f(x))^4]' = 4(f(x))^3 f'(x) = 4y^3 y' \text{ or } (y^4)' = y^3\frac{dy}{dx},$$
$$(7y)' = 7y' \text{ or } (7y)' = 7\frac{dy}{dx}.$$

The procedure we use for differentiating implicit functions is called **implicit differentiation**. Let us summarize the important steps involved in implicit differentiation.

IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x and use the Chain Rule when differentiating terms containing y.
2. Solve the resulting equation for y' or $\frac{dy}{dx}$ in terms of x and y.

Example 81 Find y' given the equation $y^5 + y + x = 0$.

Solution

$(y^5 + y + x)' = (0)'$ *(differentiate both sides)*

$(y^5)' + (y)' + (x)' = 0$ *(by the Sum Rule)*

$5y^4 y' + y' + 1 = 0$ *(by the Chain Rule)*

$y'(5y^4 + 1) = -1$ *(factorize)*

$y' = -\dfrac{1}{5y^4 + 1}$ *(isolate y')*

Example 82 Find $\frac{dy}{dx}$ given the equation $y^3 - y^2x + x^2 - 1 = 0$.

Solution

$(y^3 - y^2x + x^2 - 1)' = (0)'$ *(differentiate both sides)*

$(y^3)' - (y^2x)' + (x^2)' - (1)' = 0$ *(by the Sum Rule)*

$3y^2\frac{dy}{dx} - \left(2y\frac{dy}{dx}x + y^2\right) + 2x - 0 = 0$ *(by the Chain Rule and the Product Rule)*

$\frac{dy}{dx}(3y^2 - 2yx) = y^2 - 2x$ *(factorize)*

$\frac{dy}{dx} = \frac{y^2 - 2x}{3y^2 - 2yx}$ *(isolate $\frac{dy}{dx}$)*

Example 83 The equation $x^2 + y^2 = 4$ is given.

a. Find $\frac{dy}{dx}$ by implicit differentiation.

b. Find the slope of the tangent line to the curve at the point $(\sqrt{3}, 1)$.

c. Find the equation of the tangent line at this point.

Solution a. Differentiating both sides of the equation with respect to x, we obtain

$$(x^2 + y^2)' = (4)'$$

$$(x^2)' + (y^2)' = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad (y \neq 0).$$

$\left.\frac{dy}{dx}\right|_{(a,\, b)}$ is used for slope of the curve at the point (a, b).

b. The slope of the tangent line to the curve at the point $(\sqrt{3}, 1)$ is given by

$$m = \left.\frac{dy}{dx}\right|_{(\sqrt{3},\, 1)} = -\left.\frac{x}{y}\right|_{(\sqrt{3},\, 1)} = -\frac{\sqrt{3}}{1} = -\sqrt{3}.$$

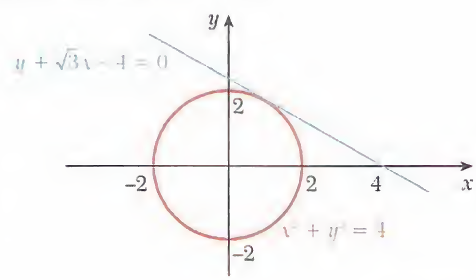
c. We can find the equation of the tangent line by using the point-slope form of the equation of a line. The slope is $m = -\sqrt{3}$ and the point is $(\sqrt{3}, 1)$. Thus,

$y - y_1 = m(x - x_1)$

$y - 1 = -\sqrt{3}(x - \sqrt{3})$

$\sqrt{3}x + y - 4 = 0.$

A sketch of this tangent line is given on the right. The line $x + \sqrt{3}y - 4 = 0$ is tangent to the graph of the equation $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1)$.



Example 84 Find the derivative with respect to x of the implicit function $\sqrt{x^2+y^2}+x^2=2$.

Solution Differentiating both sides of the given equation with respect to x, we obtain

$$\frac{d}{dx}(x^2+y^2)^{1/2}+\frac{d}{dx}(x^2)=\frac{d}{dx}(2)$$

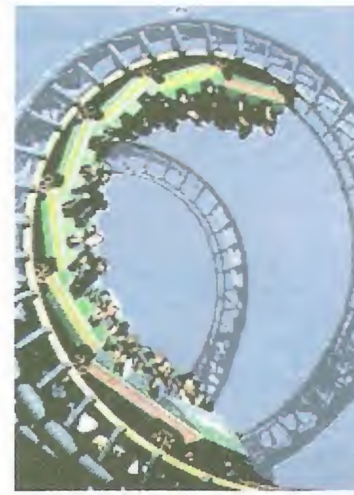
$$\frac{1}{2}(x^2+y^2)^{1/2}\frac{d}{dx}(x^2+y^2)+2x=0$$

$$\frac{1}{2}(x^2+y^2)^{-1/2}(2x+2y\frac{dy}{dx})+2x=0$$

$$2x+2y\frac{dy}{dx}=-4x(x^2+y^2)^{1/2}$$

$$2y\frac{dy}{dx}=-4x(x^2+y^2)^{1/2}-2x$$

$$\frac{dy}{dx}=\frac{-2x\sqrt{x^2+y^2}-x}{y}.$$



Check Yourself 17

1. Find $\frac{dy}{dx}$ by implicit differentiation.

   a. $x^3+x^2y+y^2=5$ b. $x^2y+xy^2=3x$

   c. $e^xe^y=1$ d. $e^x\ln y=1$

2. Find the equation of the tangent line to each curve at the given point.

   a. $x^2y^3-y^2+xy-1=0;\ (1,1)$ b. $\frac{x^2}{16}-\frac{y^2}{9}=1;\ (-5,\frac{9}{4})$

   c. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4;\ (1,3\sqrt{3})$ d. $\ln y=xy;\ (0,1)$

Answers

1. a. $\frac{-3x^2-2xy}{x^2+2y}$ b. $\frac{3-2xy-y^2}{x^2+2xy}$ c. -1 d. $-y\ln y$

2. a. $y=-\frac{3}{2}x+\frac{5}{2}$ b. $y=-\frac{5}{4}x-4$ c. $y=-\sqrt{3}x+4\sqrt{3}$ d. $y=x+1$

E. DERIVATIVES OF PARAMETRIC FUNCTIONS

Sometimes we express the variables x and y as functions of a third variable t by a pair of functions.

$$x = f(t), \quad y = g(t)$$

Functions like these are called **parametric functions**, and the variable t is called the **parameter**.

PARAMETRIC DIFFERENTIATION

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

This enables us to find the derivative of a parametric function $(\frac{dy}{dx})$ without having to eliminate the parameter t.

Example 85 Find the derivative with respect to x of the parametric curve $x = t + 2$ and $y = 2t^2 - 1$.

Solution $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4t}{1} = 4t$

Example 86 If a is a positive constant and $x = a \cos t$, $y = a \sin t$, then find $\dfrac{dy}{dx}$.

Solution $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{a \cos t}{-a \sin t} = -\cot t$

Example 87 The parametric curve is given by the equations $x = \sqrt{t+1}$ and $y = t^2 + 3t$. Find the slope of its tangent at $x = 2$.

Solution Let us begin by finding $\dfrac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2 + 3t)}{\frac{d}{dt}(\sqrt{t+1})} = \frac{2t+3}{\frac{1}{2\sqrt{t+1}}} = 2(t+1)^{1/2}(2t+3)$$

For $x = 2$, $x = \sqrt{t+1} = 2$. So, $t = 3$.

$$m = \frac{dy}{dx}\Big|_{t=3} = 2 \cdot (t+1)^{1/2} \cdot (2t+3)\Big|_{t=3} = 2 \cdot (3+1)^{1/2} \cdot (2 \cdot 3 + 3) = 2 \cdot 2 \cdot 9 = 36.$$

Example 88 The parametric equations $x = 1+e^{t-1}$, $y = t^2 + \ln t$ describe a curve in the plane. Find the tangent line to the curve at $t = 1$.

Solution First we need to find $\frac{dy}{dx}$ in terms of t:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+\frac{1}{t}}{e^{t-1}} = \frac{2t^2+1}{t} \cdot \frac{1}{e^{t-1}} = \frac{2t^2+1}{t \cdot e^{t-1}}$$

$$m = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{2t^2+1}{te^{t-1}}\right|_{t=1} = \frac{2 \cdot 1^2+1}{1 \cdot e^{1-1}} = \frac{3}{1} = 3.$$

If $t = 1$, then $x = 1 + e^{t-1} = 1 + e^{1-1} = 2$ and $y = t^2 + \ln t = 1^2 + \ln 1 = 1 + 0 = 1$.

Therefore, the line is tangent to the given curve at the point $(2, 1)$.

The equation of the line passing through $(2, 1)$ with the slope $m = 3$ is

$y - 1 = 3(x - 2)$

$y = 3x - 5$.

PARAMETRIC DIFFERENTIATION OF SECOND ORDER

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

1. Express $y' = \frac{dy}{dx}$ in terms of t.
2. Differentiate y' with respect to t.
3. Divide the result by $\frac{dx}{dt}$.

Example 89 A parametric curve is given by the equations $x = 1 + e^t$, $y = t^2e^t$.

Find the second derivative $\frac{d^2y}{dx^2}$.

Solution First, we find $\frac{dy}{dx}$ in terms of t: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^2e^t)}{\frac{d}{dt}(1+e^t)} = \frac{2te^t+t^2e^t}{e^t} = t^2+2t.$

Then, we differentiate $\frac{dy}{dx}$ with respect to t: $\frac{d}{dt}(\frac{dy}{dx}) = \frac{d}{dt}(t^2+2t) = 2t+2.$

Finally, we divide the result by $\frac{dx}{dt} = e^t$ to obtain $\frac{d^2y}{dx^2} = \frac{2t+2}{e^t}.$

Example **90** Find $\frac{d^2y}{dx^2}$, if $x = 2t - t^2$ and $y = 1 - t^3$.

Solution First, find $\frac{dy}{dx}$ in terms of t: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(1-t^3)}{\frac{d}{dt}(2t-t^2)} = \frac{-3t^2}{2-2t}$.

Then, differentiate $\frac{dy}{dx}$ with respect to t: $\frac{d}{dt}(\frac{dy}{dx}) = \frac{d}{dt}(-\frac{3t^2}{2-2t}) = \frac{-6t\cdot(2-2t)-(-3t^2)\cdot(-2)}{(2-2t)^2}$.

Finally, divide the result by $\frac{dx}{dt}$: $\frac{dx}{dt}$: $\frac{d^2y}{dx^2} = \frac{-12t+12t^2-6t^2}{(2-2t)^2\cdot(2-2t)} = \frac{6t^2-12t}{8\cdot(1-t)^3} = -\frac{3t}{4(t-1)^2}$.

Example **91** Find the second derivative of the paremetric curve given by $x = 1 + e^t$ and $y = 1 - \sin t$.

Solution First, $\frac{dy}{dx} = -\frac{\cos t}{e^t}$.

Second, $\frac{d}{dt}(\frac{dy}{dx}) = \frac{\sin t\cdot e^t - (-\cos t)e^t}{e^t} = \frac{\sin t + \cos t}{e^t}$.

Finally, $\frac{d^2y}{dx^2} = \frac{\sin t+\cos t}{e^t\cdot e^t} = \frac{\sin t+\cos t}{e^{2t}}$.

Check Yourself 18

1. Find $\frac{dy}{dx}$ for each parametric curve.

   a. $x = 2t + 3,\ y = t^2 - 1$ b. $x = 5\cos t,\ y = 5\sin t$ c. $x = \frac{t-1}{t+1},\ y = \frac{t+1}{t-1}$

2. Find $\frac{d^2y}{dx^2}$ for each parametric curve.

   a. $x = \ln t,\ y = 1 + \sin t$ b. $x = 3t^2 + 2,\ y = 2t^2 - 1$ c. $\frac{dy}{dx} = \sqrt{4+\sin^2 t},\ x = \cos 2t$

Answers

1. a. t b. $-\cot t$ c. $-\left(\frac{t+1}{t-1}\right)^2$

2. a. $t\cos t - t^2\sin t$ b. 0 c. $-\frac{1}{4\sqrt{4+\sin^2 t}}$

F. DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

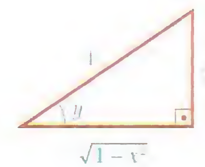
In order to find the derivatives of the inverse trigonometric functions, we can use implicit differentiation. For example, what is the derivative of Arcsin x?

Let y = Arcsin$x$, then $\sin y = x$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

Now, if we differentiate $\sin y = x$ implicitly with respect to x,

we get $\cos y \cdot \dfrac{dy}{dx} = 1$ or $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$.

$\cos y \ge 0$, since $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$. So, $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.



Therefore, $\dfrac{d}{dx}(\text{Arcsin}\, x) = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$.

We can use a similar method to find the derivative of the other inverse trigonometric functions.

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS	
$(\text{Arcsin}\, x)' = \dfrac{1}{\sqrt{1-x^2}}$	$(\text{Arcsin}\, f(x))' = \dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$
$(\text{Arccos}\, x)' = -\dfrac{1}{\sqrt{1-x^2}}$	$(\text{Arccos}\, f(x))' = -\dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$
$(\text{Arctan}\, x)' = \dfrac{1}{1+x^2}$	$(\text{Arctan}\, f(x))' = \dfrac{f'(x)}{(f(x))^2+1}$
$(\text{Arccot}\, x)' = -\dfrac{1}{1+x^2}$	$(\text{Arccot}\, f(x))' = -\dfrac{f'(x)}{(f(x))^2+1}$

Example 92 Find the derivative of the function $f(x)$ = Arcsin (x^2).

Solution $f'(x) = (\text{Arcsin}(x^2))' = \dfrac{(x^2)'}{\sqrt{1-(x^2)^2}} = \dfrac{2x}{\sqrt{1-x^4}}$

Example 93 Find the derivative of the function $f(x)$ = Arccot (e^{3x}).

Solution

$$f'(x) = (\text{Arccot}(e^{3x}))' = -\frac{(e^{3x})'}{1+(e^{3x})^2} = -\frac{3e^{3x}}{1+e^{6x}}$$

Example 94 Find the derivative of the function $f(x) = x^2 - \text{Arccos}\,(e^x)$.

Solution By the Sum Rule,

$$f'(x) = (x^2 - \text{Arccos}(e^x))' = (x^2)' - (\text{Arccos}(e^x))' = 2x - \left(-\frac{e^x}{\sqrt{1-e^{2x}}}\right) = 2x + \frac{e^x}{\sqrt{1-e^{2x}}}.$$

Example 95 Find the derivative of the function $f(x) = x \cdot \text{Arctan}\,\sqrt{x}$.

Solution By the Product Rule,

$$f'(x) = (x \cdot \text{Arctan}\,\sqrt{x})' = 1 \cdot \text{Arctan}\,\sqrt{x} + x \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \left(\frac{1}{2}x^{-1/2}\right) = \text{Arctan}\,\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}.$$

Example 96 Find the equation of the tangent line to the curve $f(x) = 2\,\text{Arccos}\,\dfrac{x}{2}$ at $x = \sqrt{3}$.

Solution

$$f'(x) = -2 \cdot \frac{\left(\dfrac{x}{2}\right)'}{\sqrt{1-\left(\dfrac{x}{2}\right)^2}} = -2 \cdot \frac{\dfrac{1}{2}}{\sqrt{1-\left(\dfrac{x}{2}\right)^2}} = -\frac{2}{\sqrt{4-x^2}}$$

The slope of the tangent line is $f'(\sqrt{3}) = -2$.

The curve passes through the point $\left(\sqrt{3}, \dfrac{2\pi}{3}\right)$ since $f'(\sqrt{3}) = \dfrac{2\pi}{3}$.

The equation of the tangent line is: $y - \dfrac{2\pi}{3} = -2(x - \sqrt{3})$ or $y = -2x + 2\sqrt{3} + \dfrac{2\pi}{3}$.

Check Yourself 19

1. Differentiate the functions.

   a. $f(x) = (e^x - \text{Arcsin}\,x)^2$

   b. $f(x) = e^{\text{Arccos}\,x}$

   c. $f(x) = (\text{Arctan}\,x)^3$

   d. $f(x) = \ln \text{Arccot}(x^2 + 5x)$

2. Find the equation of the tangent line to the curve $y = \text{Arccos}\,2x$ at the point where it crosses the y-axis.

Answers

1. a. $2(e^x - \text{Arcsin}\,x)\left(e^x - \dfrac{1}{\sqrt{1-x^2}}\right)$ b. $-\dfrac{e^{\text{Arccos}\,x}}{\sqrt{1-x^2}}$ c. $\dfrac{3(\text{Arctan}\,x)^2}{1+x^2}$

   d. $-\dfrac{2x+5}{\text{Arccot}(x^2+5x)\cdot(1+(x^2+5x)^2)}$

2. $y = -2x + \dfrac{\pi}{2}$

EXERCISES 1.3

A. Derivatives of Exponential and Logarithmic Functions

1. Differentiate the functions.

a. $f(x) = 3e^x$
b. $f(x) = e^{3x-1}$
c. $f(x) = e^{x^2-1}$
d. $f(x) = e^{-2x}$
e. $f(x) = 2^x$
f. $f(x) = (\frac{1}{3})^x$
g. $f(x) = e^x \cdot 3^x$
h. $f(x) = xe^x$
i. $f(x) = x^2 + 2e^x$
j. $f(x) = \dfrac{e^x+1}{e^x-1}$
k. $f(x) = \sqrt{e^x-1}$
l. $f(x) = (e^x + x)^{100}$
m. $f(x) = \dfrac{1}{\sqrt{e^x+1}}$
n. $f(x) = (e^x + x)(2e^x - 1)$
o. $f(x) = \dfrac{e^x+e^{-x}}{3}$
p. $f(x) = e^{\frac{1}{2x}}$
q. $f(x) = e^{\sqrt{x}+1}$
r. $f(x) = e^{\frac{\sqrt{x}-1}{\sqrt{x}+1}}$
s. $f(x) = e^{-2t} + x^2e^{x^2-1}$
t. $f(x) = \dfrac{6^x-1}{3^x+1}$
u. $f(x) = 3^{x^2+4x}$
v. $f(x) = \dfrac{5^{3x+1}}{x^2+e^x}$
w. $f(x) = \sqrt{2}^{x^2-x-1}$
x. $f(x) = 2^{e^x+2} \cdot 3^x$

2. Differentiate the functions.

a. $f(x) = 3\ln x$
b. $f(x) = \ln 4x$
c. $f(x) = 3\ln 4x$
d. $f(x) = 3\ln(2x + 1)$
e. $f(x) = \ln x^7$
f. $f(x) = \ln\sqrt{x}$
g. $f(x) = \log_3 x$
h. $f(x) = \log_{1/2} x$
i. $f(x) = x\log x$
j. $f(x) = \log_2(x^2 + 1)$
k. $f(x) = \ln(4x^2 - 6x + 3)$
l. $f(x) = \ln\dfrac{x+1}{x-1}$
m. $f(x) = \ln\sqrt{\dfrac{x-1}{x+1}}$
n. $f(x) = x^2\ln x$
o. $f(x) = \sqrt{\ln x^2}$
p. $f(x) = \sqrt{\ln x + x}$
q. $f(x) = \ln(\sqrt{x} - 1)^{-2}$
r. $f(x) = \ln(x^2 - x)$
s. $f(x) = e^x \ln x$
t. $f(x) = \dfrac{\ln(x^2-x)}{e^x}$
u. $\ln\dfrac{(x+1)(x-2)}{x-1}$
v. $f(x) = \log\dfrac{x}{x+1}$
w. $f(x) = \log_3\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$
x. $f(x) = (e^x - \log_2 x^2)^3$
y. $f(x) = \sqrt{x^2 - \log_3 e^x}$
z. $f(x) = (\log(1+e^x))^3$

3. Find the derivative of each function by using logarithmic differentiation.

a. $f(x) = (2x - 1)^7(x^4 - 3)^{11}$
b. $f(x) = x(x + 1)(x^2 + 1)$
c. $f(x) = \sqrt[3]{x^2e^{x^2-1}} \cdot (x^3 - x)^{-2}$
d. $f(x) = \dfrac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$
e. $f(x) = x^{\sqrt{x}}$
f. $f(x) = (\ln x)^{x+1}$

4. Find the equation of the tangent line to the graph of $y = e^{x^2-1}$ at the point $P(1, 1)$.

5. Find the equation of the tangent line to the curve $y = e^x + e^{-x}$ at the point $(0, 2)$.

6. Find the equation of the tangent line to the graph of $y = x^2\ln x$ at the point $(1, 0)$.

B. Derivatives of Trigonometric Functions

7. Differentiate the functions.

a. $f(x) = \sin(3x - 5)$

b. $f(x) = \cos(x^2 - 1)$

c. $f(x) = \sin x - \cos x$

d. $f(x) = 2\tan x + \sec x$

e. $f(x) = \sin x \cdot \tan x$

f. $f(x) = 2x\tan x - x\cos x$

g. $f(x) = \cos^2(2x^3 - 3x)$

h. $f(x) = \sin^3(\ln\cos 2x)$

i. $f(x) = \dfrac{\sin x + e^3}{\tan x}$

j. $f(x) = (\dfrac{1-\cos x}{1+\cos x})^{10}$

k. $f(x) = \dfrac{\cot x}{1+\sec x}$

l. $f(x) = (1 + \sec x)\cdot(1 - \cos x)$

m. $f(x) = \tan\sqrt{x^2 - x - 1}$

n. $f(x) = \dfrac{\cot(x^3-1)}{1-\sec^2(x^3-1)}$

o. $f(x) = (\dfrac{\tan(2x-1)}{2+\ln x})^3$

p. $f(x) = (\cos e^x + x\cos e^x)^2$

q. $f(x) = (e^{\sin x + \cos x} + x\cos e^x)^2$

r. $f(x) = \ln(\dfrac{1+\sin 2x}{1-\cos 2x})$

s. $f(x) = [x^2\sin(x - 1)]^5$

t. $f(x) = \sec^2(\dfrac{x^3}{x^2-1})$

u. $f(x) = \tan^2[\ln(2x + 1)]$

v. $f(x) = \csc^3\dfrac{e^x - \ln x}{x^2-1}$

8. Find the equation of the tangent line to the curve at the given point.

a. $y = e^x - \cos x + 1;\quad x = 0$

b. $y = x\cos x;\quad x = \pi$

9. For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent line?

C. Derivatives of Special Functions

10. Find the required values using the given data:

a. $f(x) = |2x - 3x^2|,\ f'(0^-) + f'\left(\left(\dfrac{2}{3}\right)^-\right) + f'(1) = ?$

b. $f(x) = x^2 + [\![x]\!] + \operatorname{sgn}(x-2),\ f'\left(\dfrac{1}{3}\right) = ?$

c. $f(x) = (2x + 3)\cdot\operatorname{sgn}(x^2 + 1),\ f'(\sqrt{2}) = ?$

d. $f(x) = \operatorname{sgn}(4x+5)\cdot[\![\dfrac{3x+1}{2}]\!],\ f'(2) = ?$

11. Given that $f(x) = |x^2 - 4| - \operatorname{sgn}(x^3 + x) + \dfrac{4x}{2x^2-1}$, find the number of different x-values for which the function is not differentiable.

D. Implicit Differentiation

12. Find $\dfrac{dy}{dx}$ for each equation below.

a. $5x - 4y = 3$

b. $xy - y - 1 = 0$

c. $x^3 + x^2 - xy = 1$

d. $\dfrac{y}{x} - 3x^2 = 5$

e. $2x^2 + 3y^2 = 12$

f. $x^2 + 5xy + y^3 = 11$

g. $x^2y^3 - xy = 8$

h. $\sqrt{xy} - 3x - y^2 = 0$

i. $e^{x+y} - e^{x-y} = 1$

j. $\ln\dfrac{x}{y} = xy - 1$

13. Find the equation of the tangent line to the given curve at the indicated point.

a. $4x^2 + 2y^2 = 12;\ (1, -2)$

b. $2x^2 + xy = 3y^2;\ (-1, -1)$

c. $e^{xy} + x^2 = 2x;\ (1, 0)$

d. $\ln(x - y) + 1 = 3x^2;\ (0, -e^{-1})$

E. Derivatives of Parametric Functions

14 . Find $\frac{dy}{dx}$ for each pair of parametric equations.

a. $x = 3t - 1$ and $y = t^2 - 2t$

b. $x = t(t + 1)$ and $y = t - \frac{1}{t}$

c. $x = t^3 - t^2 - 1$ and $y = t^2 + 3t + 1$

d. $x = \sqrt{t+1}$ and $y = t^2 + 3t$

e. $x = \sqrt[3]{t}$ and $y = \sqrt{4-t^2}$

f. $x = 4\cos t$ and $y = 5\sin t$

g. $x = t + \ln t$ and $y = 1 - e^t$

15. Find the equation of the tangent line at the given point P.

a. $x = \frac{1}{t} + t^2$, $y = t^2 - t + 1$, $P(2, 1)$

b. $x = 3t^2 + 2$, $y = 2t^4 - 1$, $P(5, 1)$

16. Find $\frac{d^2y}{dx^2}$.

a. $x = t^2 - t$, $y = t^3 + 3t + 1$

b. $x = \sqrt{t} + 1$, $y = \frac{1}{t} + 1$

c. $x = e^t - 1$, $y = \ln t$

d. $x = 2\sin^2 t$, $y = 3\cos^2 t$

F. Derivatives of Inverse Trigonometric Functions

17. Differentiate the functions.

a. $f(x) = \text{Arcsin}\,\frac{x}{2}$

b. $f(x) = \text{Arccos}\,\frac{x^2-1}{x}$

c. $f(x) = \frac{x - \text{Arcsin}\,x}{e^x}$

d. $f(x) = \ln \text{Arccos}\,e^x$

e. $f(x) = \text{Arctan}\,(x^2 + x - 1)$

f. $f(x) = \text{Arctan}\,x - \sqrt{1-x^2}$

g. $f(x) = \text{Arcsin}\,(\tan x)$

18. Find the equation of the tangent line and the normal line to $y = \text{Arcsin}\,\frac{x}{2}$ when $x = 1$.

Mixed Problems

19. Find the given order derivative by finding the first few derivatives and observing the pattern that occurs.

a. $\frac{d^{27}}{dx^{27}}(\cos x)$

b. $\frac{d^{99}}{dx^{99}}(\sin 2x)$

c. $\frac{d^{35}}{dx^{35}}(x\sin x)$

d. $\frac{d^{51}}{dx^{51}}(e^{3x+1})$

20. Find the second derivative $\frac{d^2y}{dx^2}$ of each implicit function.

a. $x^2y - 1 = 0$

b. $x^3 + y^4 = 20$

c. $y^2 + xy = 8$

d. $\sqrt[3]{x} + \sqrt[3]{y} = 1$

21. Write the equation of the line which is tangent to the curve $y = x^2 - 2|x - 1|$ at exactly two points.

22. Find the required values using the given data:

a. $f(x) = \text{Arctan}\,2x + \ln\sqrt{1+4x^2}$, $f'(1) = ?$

b. $f(x) = \sqrt{\sin\sqrt{x}}$, $f'\left(\frac{\pi^2}{16}\right) = ?$

c. $f(x) = (5x + 1)^{5x+1}$, $f'(0) = ?$

23. Given that $\frac{\pi}{2} < x < \pi$,

differentiate $f(x) = \frac{|1-\tan x|\cdot \text{sgn}(\tan x)}{[\![\frac{\cos x}{2}]\!]}$.

24. Given that $f(x) = x^3|x^2 - 2|$, find $f'(-2) + f''(1)$.

25. Given that $f(x) = \frac{x^2+1}{x^2-1}$, solve $f'(x) > 0$.

26. Given that $f(x) = (2 - \sqrt{x+2})^2$, solve $f'(x) = 0$.

27. Given the parametric equations $x = z^2 + 2z - 2$, $y = \sin(x + 2)$, $z = \ln t$, find $\left.\frac{dy}{dt}\right|_{t=1}$

CHAPTER SUMMARY

1. Introduction to Derivatives

- A tangent line to a curve is a line that touches the curve.
- The slope of a tangent line to the curve $y = f(x)$ at the point $A(a, f(a))$ is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad m = \lim_{h \to o} \frac{f(a+h) - f(a)}{h}.$$

- The problem of finding the slope of the tangent line to the graph of a function $f(x)$ at the point $A(x, f(x))$ is mathematically equivalent to the problem of finding the rate of change of $f(x)$.
- The average rate of change of $f(x)$ over the interval $[x, x + h]$ is $\dfrac{f(x+h) - f(x)}{h}$.
- The derivative of $f(x)$ with respect to x is the instantaneous rate of change of $f(x)$ and $f'(x) = \lim\limits_{h \to o} \dfrac{f(x+h) - f(x)}{h}$.
- The process of calculating the derivative of a function is called differentiation.
- If $f'(a)$ exists, then the function $f(x)$ is differentiable at a.
- If a function $f(x)$ is differentiable on the interval (a, b), then it is differentiable for every value in that interval.
- If a function $f(x)$ is differentiable at a, then
  - ➢ its graph has a non-vertical tangent line at a,
  - ➢ it is continuous at a.
- In the following cases a function is not differentiable at a given point:
  - ➢ if its graph has a corner,
  - ➢ if it is not continuous,
  - ➢ if its graph has a vertical tangent line.

2. Techniques of Differentiation

- Constant rule: $(c)' = 0$
- Power rule: $(x^n)' = n \cdot x^{n-1}$
- Constant Multiple rule: $[c \cdot f(x)]' = c \cdot f'(x)$
- Sum rule: $[f(x) + g(x)]' = f'(x) + g'(x)$
- Product rule: $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Quotient rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- Chain rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$
- General Power rule: $[(f(x))^n]' = n \cdot [f(x)]^{n-1} \cdot f'(x)$
- The derivative of the first derivative of a function is the second derivative of the function. The derivative of the second derivative is the third derivative of the function.
- The nth derivative of a function is denoted by $f^{(n)}(x)$.

3. Derivatives of Elementary Functions

- Derivatives of exponential, logarithmic and trigonometric functions:

Function	Derivative	Chain Rule
e^x	e^x	$e^{f(x)} \cdot f'(x)$
a^x	$a^x \cdot \ln a$	$a^{f(x)} \cdot \ln a \cdot f'(x)$
$\ln x$	$\dfrac{1}{x}$	$\dfrac{f'(x)}{f(x)}$
$\log_a x$	$\dfrac{1}{x \ln a}$	$\dfrac{f'(x)}{f(x) \ln a}$
$\sin x$	$\cos x$	$\cos f(x) \cdot f'(x)$
$\cos x$	$-\sin x$	$-\sin f(x) \cdot f'(x)$
$\tan x$	$\sec^2 x$	$\sec^2 f(x) \cdot f'(x)$
$\cot x$	$-\csc^2 x$	$-\csc^2 f(x) \cdot f'(x)$
$\sec x$	$\sec x \cdot \tan x$	$\sec f(x) \cdot \tan f(x) \cdot f'(x)$
$\csc x$	$-\csc x \cdot \cot x$	$-\csc f(x) \cdot \cot f(x) \cdot f'(x)$

- Logarithmic differentiation is a technique that can be used to find easily the derivatives of complicated functions involving products, quotients and powers.
- If $f(x) = |g(x)|$, then

$$f'(x) = \frac{|g(x)|}{g(x)} \cdot g'(x) = \operatorname{sgn}(g(x)) \cdot g'(x), \quad g(x) \neq 0.$$

- If $f(x) = \operatorname{sgn}(g(x))$, then

$$f'(x) = \begin{cases} 0, & g(x) \neq 0 \\ \text{does not exist}, & g(x) = 0 \end{cases}.$$

- If $f(x) = [\![g(x)]\!]$, then

$$f'(x) = \begin{cases} 0, & g(x) \notin \mathbb{Z} \\ \text{may not exist}, & g(x) \in \mathbb{Z} \end{cases}.$$

- Method of implicit differentiation:
  1. Differentiate both sides of the equation with respect to x. (Remember that y is really a function of x and the Chain Rule should be used to differentiate the terms containing y.)
  2. Solve the resulting equation for y' in terms of x and y.
- Derivatives of parametric functions:

  If $x = f(t)$ and $y = g(t)$, then

  ➢ $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$,

  ➢ $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx} = \dfrac{\frac{dy'}{dt}}{\frac{dx}{dt}}$.
- Derivatives of inverse trigonometric functions:

Function	Derivative	Chain Rule
Arcsin x	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$
Arccos x	$\dfrac{-1}{\sqrt{1-x^2}}$	$\dfrac{-f'(x)}{\sqrt{1-(f(x))^2}}$
Arctan x	$\dfrac{1}{1+x^2}$	$\dfrac{f'(x)}{1+(f(x))^2}$
Arccot x	$\dfrac{-1}{1+x^2}$	$\dfrac{-f'(x)}{1+(f(x))^2}$

Concept Check

- What is a tangent line to a curve?
- What is the expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$?
- Explain the geometrical meaning of the derivative.
- Explain the physical meaning of the derivative.
- State the limit definition of $f'(x)$.
- What do we mean when we say "when $x = 3$, the value of the function is 1"?
- What do we mean when we say "when $x = 3$, the derivative of the function is 1"?
- What do we mean when we say "f is differentiable at a"?
- What is the relation between differentiability and continuity?
- State the Power Rule.
- State the Constant Multiple Rule.
- State the Sum Rule.
- State the Product Rule.
- State the Quotient Rule.
- State the Chain Rule.
- Which rules must be applied in order to find the following derivatives?

  $[(f(x) \cdot g(x))^{2005}]'$; $\left[\dfrac{f(x)}{g(x)} + h(x) \cdot m(x)\right]'$; $(f(g(h(x))))'$
- Is it possible that the derivative of a function is equal to itself? Give an example.
- Is it possible that the derivative of a function is equal to negative of it? Give an example.
- Explain how logarithmic differentiation works.
- Explain how implicit differentiation works.

9. $f(x)=\begin{cases} x^3, & x\le 1 \\ 3x, & x>1 \end{cases}$ is given. Find $f'(1)$.

A) 0 B) 1 C) 2 D) 3 E) does not exist

10. $f(x) = 2x^2 - 3x + 1$ is given. Find $f''(1)$.

A) 8 B) 6 C) 4 D) 3 E) 1

11. $f(x) = \tan x - \cot x$ is given. Find $f'(x)$.

A) $\dfrac{4}{\sin^2 2x}$ B) $\dfrac{3}{\sin 2x}$ C) $2\tan^2 x$

D) $\tan^2 x + \cot^2 x$ E) $\sin 2x$

12. $f(x)=\sqrt{2x-1}$ is given. Find $f'(5)$.

A) $\dfrac{1}{2}$ B) $\dfrac{1}{3}$ C) $\dfrac{1}{4}$ D) $\dfrac{1}{5}$ E) $\dfrac{1}{6}$

13. Find $\dfrac{d}{dx}(\ln(\cos x))$.

A) $-\tan x$ B) $-\sec x$ C) $-\cot x$

D) $-\dfrac{1}{\sin x}$ E) $\dfrac{1}{\cos x}$

14. Find the derivative of the function $f(x) = (\sin x)^x$.

A) $(\sin x)^x \cos x$

B) $\sin x[\ln(\sin x) + \cos x]$

C) $\ln(\sin x) + x\cot x$

D) $(\sin x)^x[\ln(\sin x) + x\cot x]$

E) $(\sin x)^x(\sin x + x\cos x)$

15. If the parametric function is given by the equations $x = \sin^2\theta$, $y = \sin^2\theta$, find $\dfrac{dy}{dx}$.

A) 0 B) 1 C) -1 D) $\sin 2\theta$ E) $-\tan 2\theta$

16. The implicit function $e^x\cos y + e^y\sin x = 0$ is given. Find $\dfrac{dy}{dx}$.

A) $\tan x$ B) $\tan y$

C) $\dfrac{e^x}{e^y}\cdot(\cos y+\sin x)$ D) $\dfrac{e^x\cos y+e^y\cos x}{e^x\sin y-e^y\sin x}$

E) $\dfrac{e^x\cos y+e^y\cos x}{e^y\sin x-e^x\sin y}$

CHAPTER REVIEW TEST 1B

1. $y = 3xt - x^2t^2$ is given. Find $\frac{dy}{dt}$.

A) $3 - 2t$ B) $3t - 2t^2x$ C) $3x - 2x^2t$ D) $3x + 2x^2$ E) $3t + 2x^2t$

2. $f(x) = \frac{e^x - 1}{e^x + 1}$ is given. Find $f'(x)$.

A) $2e^x$ B) $2e^x(e^x + 1)$ C) $\frac{e^x}{e^x + 1}$ D) $\frac{2e^x}{(e^x + 1)^2}$ E) $\frac{e^x}{(e^x - 1)^2}$

3. Given that $f(1) = 3$, $\lim\limits_{x \to 1} \frac{f(x) - 3}{x - 1} = 6$ and $h(x) = x^3 \cdot f(x)$, find $h'(1)$.

A) 3 B) 6 C) 15 D) 18 E) 20

4. $f(3x - 5) = 2x^2 + x - 1$ is given. Find the value of the expression $f'(1) + f(1)$.

A) 10 B) 12 C) 14 D) 16 E) 18

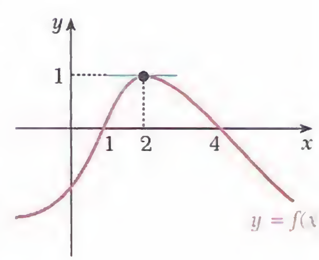
5. If the curve $y = \frac{x^2}{a}$ is tangent to the straight line with the equation $x - y = 1$, then find the value of a.

A) 5 B) 4 C) 3 D) 2 E) 1

6. If $f(x) = |2 - x| + 2$, then find the value of the expression $f'(1) + f'(3)$.

A) 0 B) 1 C) 2 D) 3 E) 4

7. The graph given on the right belongs to the function $f(x)$. If $g(x) = \frac{f(x)}{x}$, then find the slope of the tangent line to the graph of $g(x)$ at the point $x = 2$.



A) $-\frac{1}{4}$ B) $-\frac{1}{2}$ C) 2 D) 1 E) 0

8. Find the shortest distance between the curve $y = \frac{4}{x}$ and the origin (0, 0).

A) 8 B) 4 C) 2 D) $4\sqrt{2}$ E) $2\sqrt{2}$

9. $f(x) = |x-3| + \text{sgn}(x-1) + [\![x+\frac{1}{2}]\!]$ is given.

Find the derivative of the function at the point $x = \frac{1}{2}$.

A) does not exist B) 0 C) 1

D) 2 E) 18

10. Find $\frac{d^2}{dx^2}(\sin e^x)$.

A) $e^{2x}\sin e^x$ B) $e^x(\cos e^x - \sin e^x)$

C) $e^x(\cos e^x - \sin e^x)$ D) $e^x(\sin e^x - \cos e^x)$

E) $e^x(\cos e^x - e^x \sin e^x)$

11. $f(x) = \frac{\sqrt{x}}{3} - \frac{3}{\sqrt{x}}$ is given. Find $f'(9)$.

A) 9 B) 3 C) $\frac{1}{3}$ D) $\frac{1}{6}$ E) $\frac{1}{9}$

12. $f(x) = \ln\left(\frac{x^2-3x+3}{x^2-x+4}\right)$ is given. Find $\frac{df(2)}{dx}$.

A) $\frac{1}{2}$ B) e C) $\ln 3$ D) $\frac{3}{2}$ E) $\frac{5}{2}$

13. Find $\frac{d^2}{dx^2}(\sin x + \cos x)^2$.

A) $2(\cos x - \sin x)$ B) $2(\sin x - \cos x)$

C) $\sin^2 x - \cos^2 x$ D) $2\cos 2x$

E) $-4\sin 2x$

14. If the parametric curve is given by the equations $x = t^3 - 2t$, $y = t^3 - 3t$, find $\left.\frac{d^2y}{dx^2}\right|_{t=1}$.

A) 20 B) 12 C) 6 D) –20 E) –30

15. The implicit function $e^{2xy} - 4x^2 + y^3 + 7 = 0$ is given. Find $\left.\frac{dy}{dx}\right|_{t=0}$.

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $-\frac{1}{4}$ D) $-\frac{1}{3}$ E) $-\frac{2}{3}$

16. $f(x) = (x+2)^x$ is given. Find the derivative of the function at the point $x = -1$.

A) –2 B) –1 C) 0 D) 1 E) 18

1. Find $\lim\limits_{x\to 0}\dfrac{\sqrt[4]{16+x}-2}{x}$.

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{8}$ D) $\frac{1}{16}$ E) $\frac{1}{32}$

2. $f(x)=\begin{cases} x^2+2, & x\le 1 \\ 2x+1, & x>1 \end{cases}$ is given.

Find the derivative of the function at $x = 1$.

A) 0 B) 1 C) 2 D) 3 E) does not exist

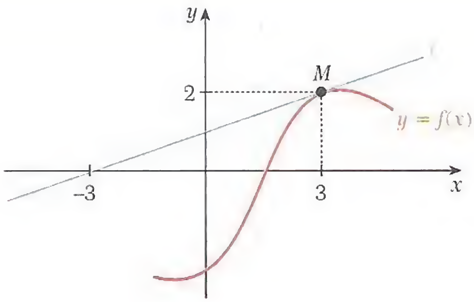
3. $f(x)=(\dfrac{x+2}{x-2})^2$ is given. Find $\dfrac{df(3)}{dx}$.

A) –300 B) –200 C) –150 D) –90 E) –40

4. $f(x) = |x^2 + 3x - 4| + [\![x^2]\!] + \operatorname{sgn}(x^2 - 1)$ is given. At which one of the following points does the derivative of the function exist?

A) 1 B) 0 C) –1 D) –2 E) –3

5.



ℓ is tangent to the curve $y = f(x)$ at the point $M(3, 2)$.

If $h(x)=\dfrac{f(x)}{x}$, find $h'(3)$.

A) $\frac{2}{9}$ B) $-\frac{5}{9}$ C) $-\frac{1}{9}$ D) $\frac{1}{3}$ E) $\frac{4}{3}$

6. $f(x + 2) = e^x \cdot g(x^2 + 1)$ and $g(1) = 5$ are given. Find the value of $f'(2)$.

A) 2 B) 3 C) 4 D) 5 E) 7

7. Find the slope of the normal line to the function $f(x) = \sin(\cos 5x)$ at the point $x=\dfrac{\pi}{10}$.

A) $-\frac{4}{5}$ B) $-\frac{1}{5}$ C) $\frac{1}{5}$ D) $\frac{2}{5}$ E) $\frac{4}{5}$

8. $f(x) = (x - 1)^2 \cdot (2x - t)$ and $f''(0) = 0$ are given. Find the value of t.

A) 4 B) 2 C) 0 D) –2 E) –4

9. Find $e^{-x} \cdot \dfrac{d^2(x^3 \cdot e^x)}{dx^2}$.

A) x^3+3x^2+3x B) x^3+3x^2+6x C) x^3+3x^2+9x

D) x^3+6x^2+6x E) x^3+9x^2+3x

10. $f(x)=\sqrt{2+\sqrt{x}}$ is given. Find $f'(4)$.

A) 1 B) 4 C) $\dfrac{1}{2}$ D) $\dfrac{1}{4}$ E) $\dfrac{1}{16}$

11. Which one of the following is correct for the tangent lines to the curve $y=\dfrac{x^3}{|x|}$ at the points $x = a$ and $x = -a$?

A) They are perpendicular to each other.

B) They are parallel to each other.

C) The angle between them is 30°.

D) They are parallel to x-axis.

E) They are parallel to y-axis.

12. $f(x)=(\arcsin\sqrt{2x})^2$ is given. Find $\dfrac{df(\frac{1}{4})}{dx}$.

A) $\dfrac{\pi^2}{4}$ B) $\dfrac{\pi^2}{2}$ C) π^2 D) π E) 2π

13. $f(x) = \ln(1-x)$ is given. Find the derivative of order n.

A) $\dfrac{(-1)^n n!}{(x-1)^n}$ B) $\dfrac{(-1)^{n+1}(n-1)!}{(x-1)^n}$

C) $\dfrac{(-1)^{n+1}(n-1)}{(x-1)^n}$ D) $\dfrac{(-1)^{2n+1}(n-1)!}{(x-1)^{2n+1}}$

E) $\dfrac{(-1)^{2n-1}(n+1)!}{(x-1)^{2n-1}}$

14. The tangent line to the curve $y = x^3$ at the point $A(2,\ 8)$ intersects the curve at another point $B(x_0,\ y_0)$. Find x_0.

A) $-\dfrac{3}{2}$ B) $\dfrac{5}{2}$ C) -3 D) -4 E) -5

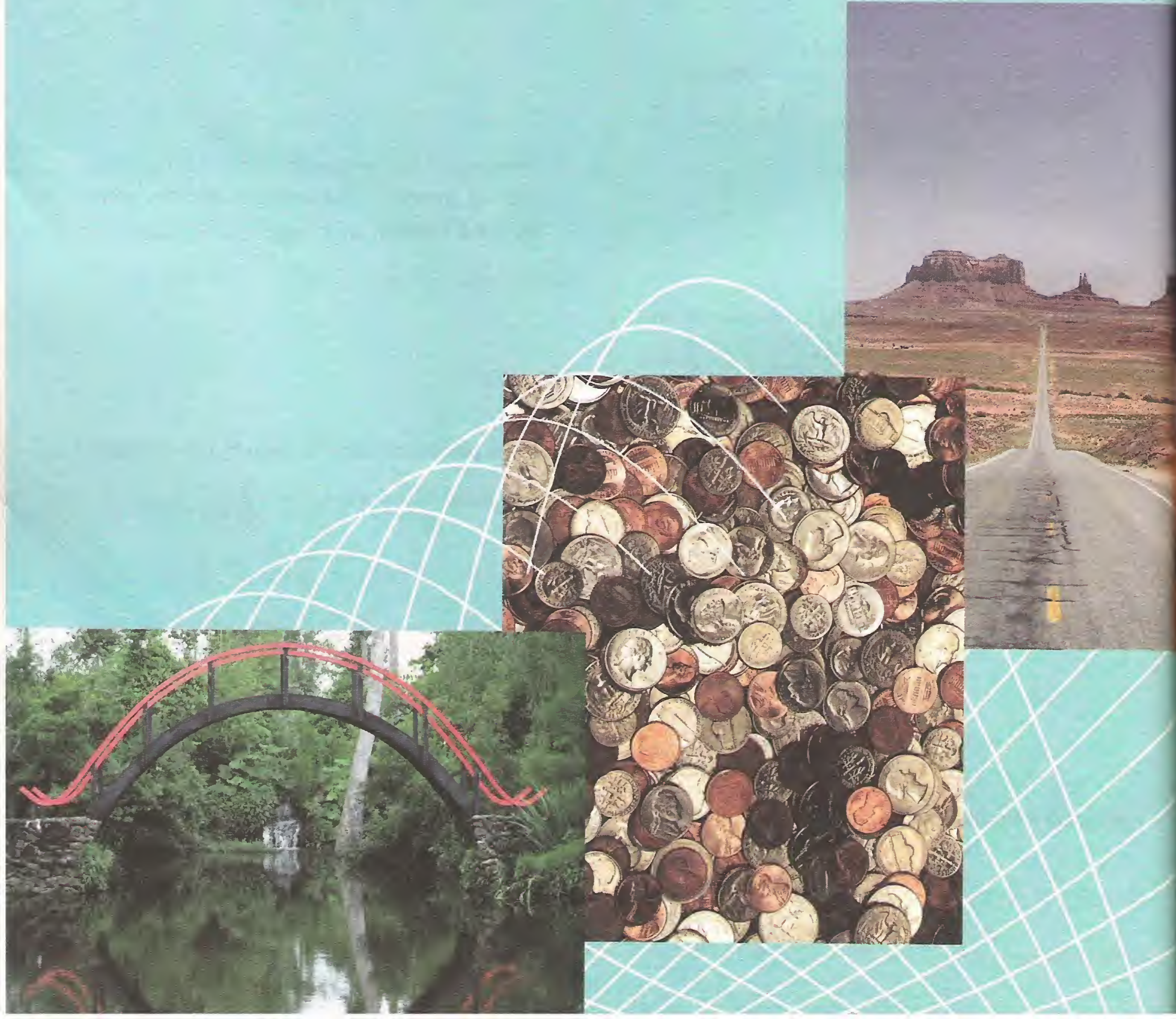
15. If the parametric curve is given by the equations $x = \sin(\ln t)$, $y = e^t \ln t$, find $\left.\dfrac{dy}{dx}\right|_{t=1}$.

A) e B) $2e$ C) $\dfrac{e}{2}$ D) $\dfrac{3}{2}$ E) $\dfrac{1}{2}$

16. The implicit function $\sin(xy) + \cos(xy) = 0$ is given. Find $\dfrac{dy}{dx}$.

A) 1 B) $-\dfrac{y}{x}$ C) $2xy$

D) $y\cos(xy)$ E) $x\cos x - y\cos y$



CHAPTER 2

APPLICATIONS OF THE DERIVATIVE

1 L'HOSPITAL'S RULE

A. THE INDETERMINATE FORM $\frac{0}{0}$

GUILLAUME DE L'HOSPITAL (1661-1704)

French mathematician solved a difficult problem posed by Pascal at age 15. He published the first book ever on differential calculus "L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes" (1696). In this book, L'Hospital introduced L'Hospital's rule. Within the book, L'Hospital thanks Bernoulli for his help. An earlier letter by John Bernoulli gives both the rule and its proof, so it seems likely that it was Bernoulli who discovered the rule. L'Hospital's name is spelled both "L'Hospital" and "L'Hôpital", the two being equivalent in French spelling.

Let us consider the following limit where both $f(x)$ and $g(x)$ approach to zero as $x \to a$:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

If we substitute $x = a$ in this limit, we obtain a fraction of the form $\frac{0}{0}$, which is a meaningless algebraic expression. This limit may or may not exist and is called an **indeterminate form** $\frac{0}{0}$.

From earlier studies you have learned to calculate such limits by using the limit theorems. In this section, we will discuss a very powerful method known as **L'Hospital's Rule**. This rule gives a connection between derivatives and limits of the indeterminate form $\frac{0}{0}$.

L'HOSPITAL'S RULE

Let the functions f and g be differentiable on an open interval that contains the point a.

Suppose that $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ and $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists. Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

Example 1 Find $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

Solution Since $\lim\limits_{x \to 0} \sin x = 0$ and $\lim\limits_{x \to 0} x = 0$, we can apply L'Hospital's Rule.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{(x)'} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

Note

Using L'Hospital's Rule, differentiate both the numerator and the denominator seperately. Do not apply the Quotient Rule.

Example 2 Find $\lim\limits_{x \to 4} \dfrac{x-4}{x^2-4}$.

Solution If we attempted to use L'Hospital's Rule, we would get

$$\lim_{x \to 4} \frac{x-4}{x^2-4} = \lim_{x \to 4} \frac{(x-4)'}{(x^2-4)'} = \lim_{x \to 4} \frac{1}{2x} = \frac{1}{8}.$$

This is wrong!

Since $\lim\limits_{x \to 4} \dfrac{x-4}{x^2-4}$ does not give the indeterminate form $\dfrac{0}{0}$, we cannot apply L'Hospital's Rule here. Let us find the correct answer:

$$\lim_{x \to 4} \frac{x-4}{x^2-4} = \frac{4-4}{4^2-4} = \frac{0}{12} = 0$$

Note

Before applying L'Hospital's Rule, verify that we have the indeterminate form $\dfrac{0}{0}$.

Example 3 Find $\lim\limits_{x \to -1} \dfrac{x^3+x+2}{x+1}$.

Solution

$$\lim_{x \to -1} \frac{x^3+x+2}{x+1} = \frac{(-1)^3+(-1)+2}{-1+1} = \frac{0}{0} \qquad \left(\frac{0}{0} \textit{ form; apply the rule}\right)$$

$$\lim_{x \to -1} \frac{x^3+x+2}{x+1} = \lim_{x \to -1} \frac{(x^3+x+2)'}{(x+1)'} = \lim_{x \to -1} \frac{3x^2+1}{1} = 3 \cdot (-1)^2 + 1 = 4$$

Example 4 Find $\lim\limits_{x \to 1} \dfrac{\ln x}{x^2-1}$.

Solution

$$\lim_{x \to 1} \frac{\ln x}{x^2-1} = \frac{0}{0} \qquad \left(\frac{0}{0} \textit{ form; apply the rule}\right)$$

$$\lim_{x \to 1} \frac{\ln x}{x^2-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x^2-1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$$

Example 5 Find $\lim\limits_{x\to -1}\dfrac{\sqrt{2x+3}-1}{\sqrt{x+5}-2}$.

Solution We have the indeterminate form $\dfrac{0}{0}$. So, we can use L'Hospital's Rule:

$$\lim_{x\to -1}\frac{\sqrt{2x+3}-1}{\sqrt{x+5}-2}=\lim_{x\to -1}\frac{(\sqrt{2x+3}-1)'}{(\sqrt{x+5}-2)'}=\lim_{x\to -1}\frac{\dfrac{2}{2\sqrt{2x+3}}}{\dfrac{1}{2\sqrt{x+5}}}=\lim_{x\to -1}\frac{2\sqrt{x+5}}{\sqrt{2x+3}}=\frac{2\cdot\sqrt{4}}{\sqrt{1}}=4.$$

Note

If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ is still indeterminate form $\dfrac{0}{0}$, we use L'Hospital's Rule again.

That gives $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=\lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$.

In fact, whenever L'Hospital's Rule gives $\dfrac{0}{0}$, we can apply it again until we get a different result.

Example 6 Find $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^2}$.

Solution

$$\lim_{x\to 0}\frac{x-\sin x}{x^2}=\frac{0}{0} \qquad \left(\frac{0}{0}\textit{ form; apply the rule}\right)$$

$$\lim_{x\to 0}\frac{x-\sin x}{x^2}=\lim_{x\to 0}\frac{(x-\sin x)'}{(x^2)'}=\lim_{x\to 0}\frac{1-\cos x}{2x}=\frac{1-\cos 0}{2\cdot 0}=\frac{0}{0} \qquad \left(\frac{0}{0}\textit{ form; apply the rule again}\right)$$

$$\lim_{x\to 0}\frac{1-\cos x}{2x}=\lim_{x\to 0}\frac{(1-\cos x)'}{(2x)'}=\lim_{x\to 0}\frac{\sin x}{2}=\frac{0}{2}=0$$

Example 7 Find $\lim\limits_{x\to 1}\dfrac{e^x-ex}{(x-1)^2}$.

Solution

$$\lim_{x\to 1}\frac{e^x-ex}{(x-1)^2}=\frac{0}{0} \qquad \left(\frac{0}{0}\textit{ form; apply the rule}\right)$$

$$\lim_{x\to 1}\frac{e^x-ex}{(x-1)^2}=\lim_{x\to 1}\frac{(e^x-ex)'}{[(x-1)^2]'}=\lim_{x\to 1}\frac{e^x-e}{2(x-1)} \qquad \left(\frac{0}{0}\textit{ form; apply the rule again}\right)$$

$$\lim_{x\to 1}\frac{e^x-e}{2(x-1)}=\lim_{x\to 1}\frac{(e^x-e)'}{[2(x-1)]'}=\lim_{x\to 1}\frac{e^x}{2}=\frac{e}{2}$$

Check Yourself 1

Find the following limits:

1. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}$
2. $\lim\limits_{x\to 4}\dfrac{x^2-8\sqrt{x}}{x-4}$
3. $\lim\limits_{x\to 0}\dfrac{e^x-1}{\ln\sqrt{x+1}}$
4. $\lim\limits_{x\to \pi/2}\dfrac{\sin x-1}{\cos 2x+1}$

Answers

1. 3 2. 6 3. 2 4. –1/4

B. THE INDETERMINATE FORM $\frac{\infty}{\infty}$

L'Hospital's Rule is also valid for the indeterminate form $\frac{\infty}{\infty}$. It is expressed as follows:

Suppose that $\lim\limits_{x \to a} f(x) = \pm\infty$, $\lim\limits_{x \to a} g(x) = \pm\infty$ and $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

Then, $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

Example 8 Find $\lim\limits_{x\to\infty}\frac{x^2-3x+5}{7+2x-3x^2}$.

Solution $\lim\limits_{x\to\infty}\frac{x^2-3x+5}{7+2x-3x^2} = \frac{\infty}{-\infty}$ *($\frac{\infty}{\infty}$ form: apply the rule)*

$\lim\limits_{x\to\infty}\frac{x^2-3x+5}{7+2x-3x^2} = \lim\limits_{x\to\infty}\frac{(x^2-3x+5)'}{(7+2x-3x^2)'} = \lim\limits_{x\to\infty}\frac{2x-3}{2-6x}$ *(still $\frac{\infty}{\infty}$ form: apply the rule)*

$\lim\limits_{x\to\infty}\frac{2x-3}{2-6x} = \lim\limits_{x\to\infty}\frac{(2x-3)'}{(2-6x)'} = \lim\limits_{x\to\infty}\frac{2}{-6} = -\frac{1}{3}$

Example 9 Find $\lim\limits_{x\to\infty}\frac{e^x+2x}{e^{2x}-3}$.

Solution $\lim\limits_{x\to\infty}\frac{e^x+2x}{e^{2x}-3} = \frac{\infty}{\infty}$ *($\frac{\infty}{\infty}$ form: apply the rule)*

$\lim\limits_{x\to\infty}\frac{e^x+2x}{e^{2x}-3} = \lim\limits_{x\to\infty}\frac{(e^x+2x)'}{(e^{2x}-3)'} = \lim\limits_{x\to\infty}\frac{e^x+2}{2\cdot e^{2x}}$ *(still $\frac{\infty}{\infty}$ form: apply the rule)*

$\lim\limits_{x\to\infty}\frac{e^x+2}{2\cdot e^{2x}} = \lim\limits_{x\to\infty}\frac{(e^x+2)'}{(2\cdot e^{2x})'} = \lim\limits_{x\to\infty}\frac{e^x}{4\cdot e^{2x}} = \lim\limits_{x\to\infty}\frac{1}{4\cdot e^x} = \frac{1}{\infty} = 0$

Example 10 Find $\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{\ln x}$.

Solution When $x \to 0^+$, $\frac{1}{x} \to \infty$ and $\ln x \to -\infty$. So, we can apply L' Hôs pital's Rule.

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{\ln x} = \lim_{x\to 0^+}\frac{(\frac{1}{x})'}{(\ln x)'} = \lim_{x\to 0^+}\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to 0^+} -\frac{1}{x} = -\infty.$$

Note

L'Hospital's Rule cannot be applied directly to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. But it may be possible to convert them into the form $\frac{0}{0}$ or into the form $\frac{\infty}{\infty}$.

Example 11 Find $\lim\limits_{x\to\infty} x \sin\frac{1}{x}$.

Solution This limit leads to the form $\infty \cdot 0$, but we can change it to the form $\frac{0}{0}$ by writing $x = \frac{1}{\frac{1}{x}}$.

$f \cdot g = \dfrac{f}{\frac{1}{g}}$ or $f \cdot g = \dfrac{g}{\frac{1}{f}}$

$$\lim_{x\to\infty} x \cdot \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \frac{0}{0}$$

Since we have the form $\frac{0}{0}$, we can apply L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{(\sin\frac{1}{x})'}{(\frac{1}{x})'} = \lim_{x\to\infty} \frac{(-\frac{1}{x^2}) \cdot \cos\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \cos\frac{1}{x} = 1$$

Example 12 Find $\lim\limits_{x\to 0}(\frac{1}{x} - \frac{1}{e^x - 1})$.

Solution We have the indeterminate form $\infty - \infty$, but we write

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{e^x - 1}) = \lim_{x\to 0} \frac{e^x - 1 - x}{x \cdot (e^x - 1)} = \frac{0}{0} \quad (\tfrac{0}{0} \textit{ form; apply the rule})$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x \cdot (e^x - 1)} = \lim_{x\to 0} \frac{(e^x - 1 - x)'}{[x \cdot (e^x - 1)]'} = \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + x \cdot e^x} = \frac{0}{0} \quad (\tfrac{0}{0} \textit{ form; apply the rule again})$$

$$\lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + x \cdot e^x} = \lim_{x\to 0} \frac{(e^x - 1)'}{(e^x - 1 + x \cdot e^x)'} = \lim_{x\to 0} \frac{e^x}{e^x + e^x + x \cdot e^x} = \frac{1}{2}.$$

Check Yourself 2

Find the following limits:

1. $\lim\limits_{x\to\infty} \dfrac{2x^2 - 5x + 7}{3x + 4}$
2. $\lim\limits_{x\to\infty} \dfrac{x^3 - 3x + 5}{e^x}$
3. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{\ln\sqrt{x}}$
4. $\lim\limits_{x\to\infty} xe^{-x}$
5. $\lim\limits_{x\to 0}(\dfrac{1}{x} - \dfrac{1}{\sin x})$

Answers

1. ∞ 2. 0 3. ∞ 4. 0 5. 0

EXERCISES 2.1

A. The Indeterminate Form $\frac{0}{0}$

1. Find the following limits:

a. $\lim\limits_{x\to 2}\dfrac{x^2-4x+4}{x^2+x-6}$ b. $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{3}}{2x}$

c. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x}-1}{x}$ d. $\lim\limits_{x\to -1}\dfrac{4x^3+4}{5x^2+6x+1}$

e. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ f. $\lim\limits_{x\to 1}\dfrac{1-x}{\ln x}$

g. $\lim\limits_{x\to 2}\dfrac{x^2-\sqrt{8x}}{x-2}$ h. $\lim\limits_{x\to -2}\dfrac{\sin(2x+4)}{x^2-x-6}$

i. $\lim\limits_{x\to 2}\dfrac{2-\sqrt[3]{x+6}}{x^2-4}$ j. $\lim\limits_{x\to 3}\dfrac{\sqrt[3]{3}-\sqrt[3]{x}}{\sqrt{x}-\sqrt{3}}$

k. $\lim\limits_{x\to 1/2}\dfrac{2x^2+3x-2}{6x-3}$ l. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

m. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\ln(x^2+x+1)}$ n. $\lim\limits_{x\to 2}\dfrac{\tan(x^2-4)}{4-x^2}$

o. $\lim\limits_{x\to 0}\dfrac{3^x-7^x}{x}$ p. $\lim\limits_{x\to \pi}\dfrac{\cos\frac{5x}{6}+\sin\frac{x}{3}}{x-\pi}$

q. $\lim\limits_{x\to 0}\dfrac{\arctan 3x}{\arctan 4x}$ r. $\lim\limits_{x\to \pi/2}\dfrac{e^{\cos x}-1}{\cos x}$

s. $\lim\limits_{x\to 8}\dfrac{x^{2/3}-4}{x-8}$ t. $\lim\limits_{x\to 2}\dfrac{\arctan(\frac{x}{2})-\frac{\pi}{4}}{x-2}$

B. The Indeterminate Form $\frac{\infty}{\infty}$

2. Find the following limits:

a. $\lim\limits_{x\to \infty}\dfrac{x^2+x+1}{3x^3+4}$ b. $\lim\limits_{x\to -\infty}\dfrac{9-x^2}{x^2-2x-3}$

c. $\lim\limits_{x\to \infty}\dfrac{e^x-1}{e^x+1}$ d. $\lim\limits_{x\to \infty}\dfrac{\sqrt{x^2+4}}{x}$

e. $\lim\limits_{x\to \infty}\dfrac{\ln x}{x^5}$ f. $\lim\limits_{x\to \infty}\dfrac{e^x}{x^2+x}$

g. $\lim\limits_{x\to \infty}\dfrac{\ln(1+e^x)}{2x}$ h. $\lim\limits_{x\to \infty}\dfrac{\ln x}{\sqrt{x}}$

i. $\lim\limits_{x\to 0}\dfrac{\cot 3x}{\cot 5x}$ j. $\lim\limits_{x\to \infty}\dfrac{x^2+\cos x}{3x^2}$

Mixed Problems

3. Find the following limits:

a. $\lim\limits_{x\to 1}\dfrac{x^3+x-2}{2x^3-3x+1}$ b. $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\tan x}{1+\tan x}$

c. $\lim\limits_{x\to \infty}x\ln(1+\dfrac{1}{x})$ d. $\lim\limits_{x\to \frac{\pi}{2}}(\tan x-\sec x)$

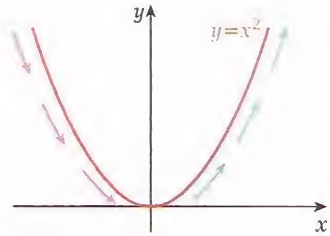
e. $\lim\limits_{x\to 0}\dfrac{\sin^6 x}{3x}$ f. $\lim\limits_{x\to 1}\dfrac{x-1}{2}\tan\dfrac{\pi x}{2}$

g. $\lim\limits_{x\to 0}(\dfrac{1}{x}-\csc x)$ h. $\lim\limits_{x\to \infty}(\sqrt{x^2+3x}-x)$

2 APPLICATIONS OF THE FIRST DERIVATIVES

A. INTERVALS OF INCREASE AND DECREASE

In this section, we will first briefly review the increasing and decreasing functions and then discuss the relationship between the sign of the derivative of a function and the increasing and decreasing behavior of the function.



the graph of $f(x) = x^2$

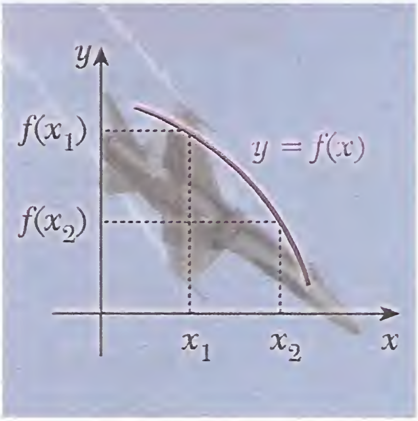
Recall the graph of the function $f(x) = x^2$. As we move from left to right along its graph, we see that the graph of f falls for $x < 0$ and rises for $x > 0$. The function f is said to be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

Definition

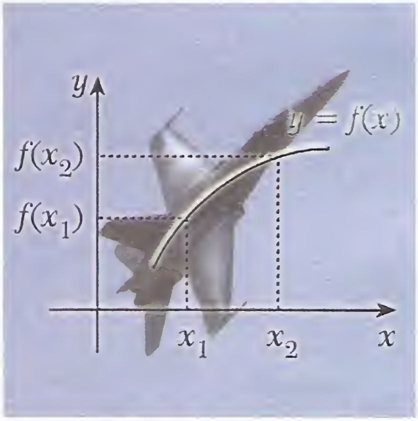
increasing and decreasing functions

A function f is **increasing** on an interval I if $f(x)$ increases as x increases on I. That is, for any $x_1 < x_2$ on I, $f(x_1) < f(x_2)$.

Similarly, f is **decreasing** on an interval I if $f(x)$ decreases as x increases on I. That is, for any $x_1 < x_2$ in I, $f(x_1) > f(x_2)$.



a decreasing function

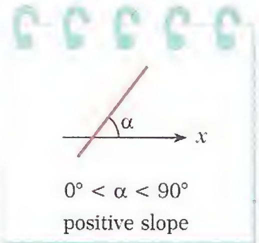


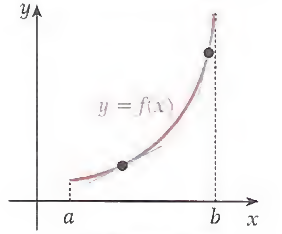
an increasing function

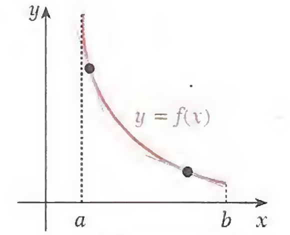
Note

We refer to a function as increasing or decreasing only on intervals, not at particular points.

We now learn how the first derivative can be used to determine where the function is increasing or decreasing. Let us look at the following graphs.







Observe that the function f is increasing on the interval (a, b) and the tangent lines to the graph of f have positive slope on that interval. We know that the slope of each tangent line is given by the derivative $f'(x)$. Thus, $f'(x)$ must be positive on (a, b).

Similarly, we expect to see a decreasing function when $f'(x)$ is negative. These observations lead to the following important theorem.

Theorem

Let $f(x)$ be a differentiable function on the interval I.

a. If $f'(x) > 0$ for all the values of x on I, then $f(x)$ is increasing on the interval I.

b. If $f'(x) < 0$ for all the values of x on I, then $f(x)$ is decreasing on the interval I.

Note

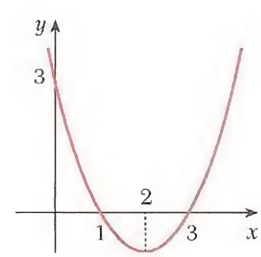
According to the theorem above, when we are asked to determine the intervals of increase and decrease for a given function, we must examine the sign of the derivative of the function. To do this, we shall construct the sign chart of the first derivative. We assume that you are familiar with constructing the sign chart of a function from your earlier studies.

Example 13 Find the intervals where the function $f(x) = x^2 - 4x + 3$ is increasing and where it is decreasing.

Solution Let us construct the sign chart of $f'(x)$.

$f'(x) = 2x - 4$ and $x = 2$ is a root of $f'(x) = 0$

x	$-\infty$	2	$+\infty$
$f'(x)$	$-$	0	$+$
$f(x)$	decreasing (↘)		increasing (↗)



From the chart, $f'(x) > 0$ when $x > 2$ and $f'(x) < 0$ when $x < 2$.

So, f is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

It would also be true to say that f is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.

Note

As in the previous example, we use an "up arrow" (↗) for the intervals where the function is increasing and a "down arrow" (↘) for the intervals where the function is decreasing.

Example 14 For what values of x is the function $f(x) = (x-1)^3$ either increasing or decreasing?

Solution $f'(x) = 3(x-1)^2$.

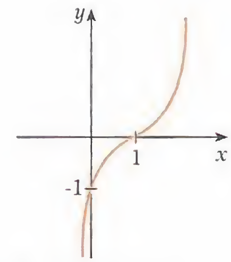
$f'(x) \geq 0$ for any of the values of x because $(x-1)^2 > 0$.

We conclude that f is always increasing for all the values of x.

The graph of f is shown in the figure.

Note that when $x = 1$ we have $f'(x) = 0$.

But that does not affect the increase since it is just a point.



Example 15 Determine where $f(x) = e^{x^3-3x}$ is increasing.

Solution $f'(x) = e^{x^3-3x} \cdot (3x^2-3)$

Since $e^{x^3-3x} > 0$ for all the values of x, it is enough to check the sign of $3x^2-3$.

$3x^2 - 3 = 0 \Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1$ and $x = 1$.

x	$-\infty$	-1		1	$+\infty$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗		↘		↗

The chart suggests that $f(x)$ is increasing for $x < -1$ and $x > 1$.

Example 16 For what values of a is the function
$f(x) = ax^3 - 2x^2 + 2x - 3$ increasing for all real numbers?

Solution Since f is increasing for all real numbers, $f'(x) > 0$.

$f'(x) = 3ax^2 - 4x + 2 > 0$

This is possible only if $3a > 0$ and $\Delta < 0$

$3a > 0$. So, $a > 0$. (1)

$\Delta = (-4)^2 - 4 \cdot 3a \cdot 2 = 16 - 24a < 0$. So, $a > \frac{2}{3}$. (2)

By (1) and (2), we have $a > \frac{2}{3}$.

Given $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$.

Example 17 For what values of m is $f(x)=\dfrac{mx-2}{x+3}$ always decreasing in its domain?

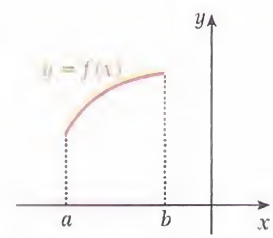
Solution Since f is always decreasing, $f'(x) < 0$ for all the values of x except $x = -3$.

$$f'(x)=\frac{m(x+3)-(mx-2)}{(x+3)^2}=\frac{3m+2}{(x+3)^2}<0.$$

Then, we have $3m + 2 < 0$ because $(x + 3)^2$ is always positive.

Thus, $m<-\dfrac{2}{3}$.

EXAMPLE 18 The graph of the function f is given on the interval (a, b). State whether each of the following functions is increasing or decreasing on (a, b).



a. $g(x) = x^2 - f(x)$ b. $h(x) = f^2(x)$ c. $m(x)=\dfrac{f(x)}{x}$

Solution From the graph, we conclude that $x < 0$ and $f(x) > 0$ on (a, b). Also, we have $f'(x) > 0$ because f is increasing on (a, b). Now let us find the derivative of each function.

a. $g'(x) = 2x - f'(x) < 0$. So g is decreasing on (a, b).

b. $h'(x) = 2 \cdot f(x) \cdot f'(x) > 0$. So h is increasing on (a, b).

c. $m'(x) = \dfrac{f'(x)\cdot x-f(x)}{x^2}<0.$ So m is decreasing on (a, b).

Check Yourself 3

1. Find the intervals where each function is increasing or decreasing.

   a. $f(x) = x^3 - 3x^2 + 6$ b. $f(x)=\dfrac{1}{3x+4}$ c. $f(x) = \ln x$

2. The function $f(x)=\dfrac{kx+1}{x+1}$ is always increasing in its domain. Find k.

Answers

1. a. increasing on $(-\infty, 0)$ and $(2, \infty)$, decreasing on $(0, 2)$

   b. decreasing on $\left(-\infty,-\dfrac{4}{3}\right)$ and $\left(-\dfrac{4}{3},\infty\right)$

   c. increasing on $(0, \infty)$

2. $k > 1$.

B. MAXIMUM AND MINIMUM VALUES

1. Absolute and Local Maximum and Minimum

In many applications we need to find the largest or the smallest value of a specified quantity. Here are a few examples:

- What is the shape of a container that minimizes the manufacturing costs?
- At what temperature does a certain chemical reaction proceed most rapidly?
- Which path requires the least time to travel?

These problems can be reduced to finding the maximum or minimum value of a function. Let us first explain what we mean by maximum and minimum values.

Definition

absolute maximum and minimum

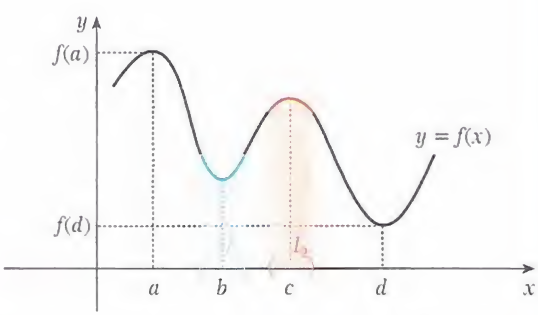
A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all the values of x in its domain. Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all the values of x in its domain.

Note

Here is some terminology: If a function f has a maximum (or minimum) at $x = c$, then we say that $f(c)$ is a maximum (or minimum) value of f and $(c, f(c))$ is a maximum (or minimum) point of f.

An **extremum** of a function is either a maximum or minimum value of that function.

The figure on the right shows the graph of a function f with absolute maximum at $x = a$ and absolute minimum at $x = d$. Note that $(a, f(a))$ is the highest point on the graph and $(d, f(d))$ is the lowest point.



In the same graph, if we consider only the values of x sufficiently near b (for example, in the interval I_1), then $f(b)$ is the smallest of those values of $f(x)$. In other words, no nearby points on the graph of f are lower than the point $(b, f(b))$. To define such points, we use the word "*local*". So, we say that the function f has a **local minimum** at the point $x = b$.

Similarly, f has a **local maximum** at $x = c$ because $f(c)$ is the largest value of $f(x)$ in the interval I_2. We see that no nearby points on the graph are higher than the point $(c, f(c))$. We now state the formal definition:

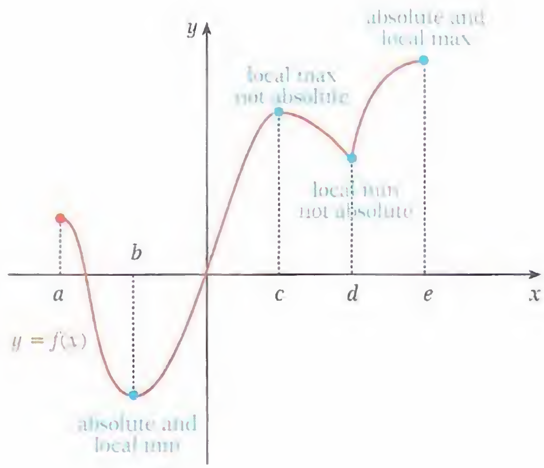
Definition

local maximum and minimum

A function f has a **local maximum** at c if $f(c) \geq f(x)$ for all the values of x in an interval I containing c.

Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ for all the values of x in an interval I containing c.

The figure on the right illustrates some local and absolute extrema of a function f with the domain $[a, e]$. We see that f has a local maximum at $x = c$, and a local minimum at $x = b$ and $x = d$. Also, f has an absolute minimum at $x = b$ and an absolute maximum at $x = e$. Observe that the absolute minimum is also local, but the absolute maximum is not local because it occurs at the endpoint $x = e$.

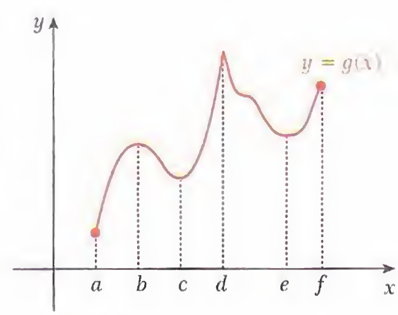


Note

1. A function has at most one absolute maximum and one absolute minimum. But it may have more than one local maximum or minimum.
2. An absolute extremum of a function is either a local extremum or an endpoint.

Check Yourself 4

1. Explain the difference between an absolute maximum and a local maximum.
2. The graph of a function with the domain $[a, f]$ is given on the right. For each of the points from a to f, state whether the function has a local maximum or minimum, or an absolute maximum or minimum.



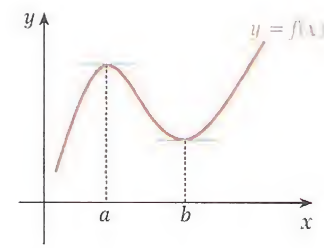
Answers

1. absolute max.: $f(c) \geq f(x)$ for all the values of x in the domain of f.

   local max.: $f(c) \geq f(x)$ for all the values of x in an interval I containing c.
2. local max. at $x = b$, $x = d$
   local min. at $x = c$, $x = e$
   absolute max. at $x = d$
   absolute min. at $x = a$.

2. Finding the Local Extrema

We now learn how the first derivative can be used to locate the local extrema. We first consider the functions that have derivatives at the local extremum points. The figure on the right shows the graph of a function f with a local maximum at $x = a$ and a local minimum at $x = b$. Observe that the tangent lines to the graph at these points are horizontal (parallel to the x-axis) and therefore each has slope 0. Remember that the slope of the tangent line is given by the derivative. So, we say that



$$f'(a) = 0, f'(b) = 0.$$

This analysis reveals an important characteristic of the local extrema of a differentiable function.

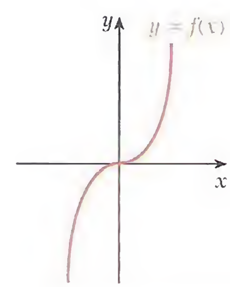
Theorem

If f has a local extremum at c, and $f'(c)$ exists, then $f'(c) = 0$.

Note

The converse of this theorem is not true in general.

That is, when $f'(c) = 0$, f does not necessarily have a maximum or minimum at $x = c$. For example, consider the function $f(x) = x^3$. Here, $f'(x) = 3x^2$, so $f'(0) = 0$. But, f has neither a local maximum nor a local minimum at $x = 0$.



Example 19 The function $f(x) = 2x^3 - mx + 5$ has a local minimum at $x = 1$. Find m.

Solution Since $f(x)$ is a polynomial function, it is differentiable everywhere. By the theorem above, we have $f'(1) = 0$.

$f'(x) = 6x^2 - m$

$f'(1) = 0$

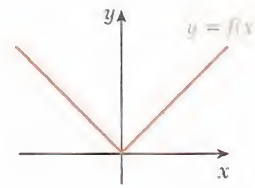
$6 \cdot 1^2 - m = 0$

$m = 6$

Example **20** Find the local extrema of the function $f(x) = |x|$.

Solution Let us plot the graph of f.

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



We see that f has a local minimum at $x = 0$.

But there is no horizontal tangent there.

In fact, there is no tangent line at all since $f'(x)$ is not defined at $x = 0$.

In some books, an extremum that occurs at a point where the derivative does not exist is called a **singular point**.

Note

The function $f(x) = |x|$ shows that a local extremum of a function may exist at which the derivative does not exist. As a consequence, we say that the local extrema of any function f occurs at the points c where $f'(c) = 0$ or $f'(c)$ does not exist. Such points are given a special name.

Definition

critical point

The value c in the domain of f is called a **critical point** if either

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist.

Example **21** Find the critical points of $f(x) = 2x^3 - 9x^2 + 12x - 7$.

Solution The derivative of f is $f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$.

Since $f'(x)$ is defined for all the values of x, the only critical points are the roots of $f'(x) = 0$.

Therefore, $x = 1$ and $x = 2$.

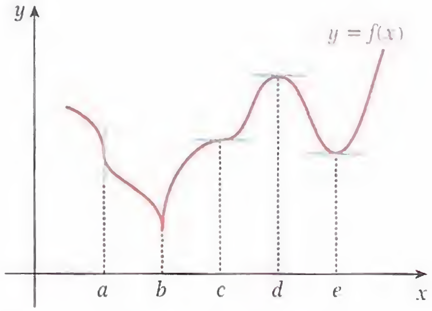
Example **22** Find the critical points of $f(x) = \dfrac{x^2}{x-1}$.

Solution The Quotient Rule gives

$$f'(x) = \frac{2x(x-1) - x^2 \cdot 1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Since $x = 0$ and $x = 2$ are the roots of $f'(x) = 0$, they are critical points. Next, observe that $f'(x)$ does not exist when $x = 1$. However, since f is not defined at that point, the point $x = 1$ is not a critical point.

Example 23 In the figure on the right the graph of a function f with critical points at $x = a, b, c, d$, and e are shown.



a. State why these points are critical.

b. Classify each of them as a local maximum, a local minimum, or neither.

Solution a. Observe that there are horizontal tangents at the points $x = c$, d, and e, so $f'(x) = 0$ at these points. Next, $f'(x)$ does not exist at $x = a$ because the tangent line at this point is vertical. Finally, since there is a corner at $x = b$, $f'(x)$ does not exist there.

b. From the graph of f, we say that f has a local maximum at $x = d$, and a local minimum at $x = b$ and $x = e$. Note that $f'(c) = 0$ and $f'(a)$ does not exist, and f has no local extrema at these points. We conclude that not every critical point gives rise to a local extrema.

Check Yourself 5

Find the critical points of the following functions.

1. $f(x) = x^3 - 3x + 4$ 2. $f(x) = 1 - \sqrt{x}$ 3. $f(x) = x^2 \ln x$

Answers

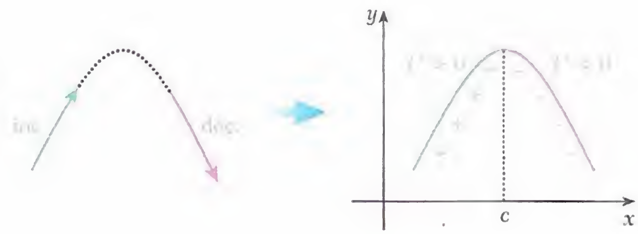
1. $-1, 1$ 2. 0 3. $\frac{1}{\sqrt{e}}$

3. The First Derivative Test

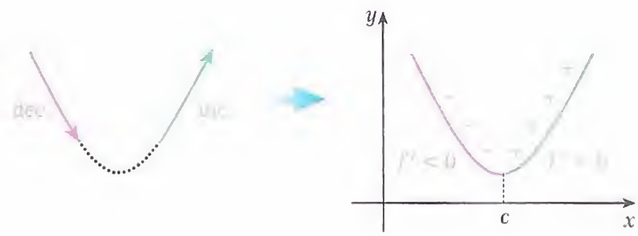
So far we have learned that any extremum of a function f must occur at any critical point of f. In the previous example we have seen that not every critical point is a maximum or a minimum. Therefore, we need a test that helps us classify critical points as local maximum, local minimum, or neither.

Suppose that the function f is continuous at c and that f is defined on some open interval containing c.

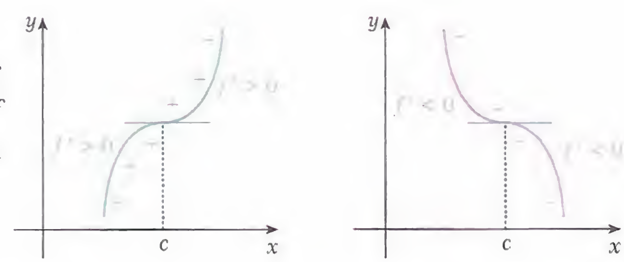
If f is increasing on the left of c and decreasing on the right, then f should have a local maximum at $x = c$.



If f is decreasing on the left of c and increasing on the right, then f should have a local minimum at $x = c$.



If f is increasing on both sides or decreasing on both sides, then f should have neither a local maximum nor a local minimum at $x = c$.



Moreover, we know that $f(x)$ is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$. These observations are the basis of the following test.

THE FIRST DERIVATIVE TEST

Let c be a critical point of a function $f(x)$.

1. If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum at c.
2. If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum at c.
3. If $f'(x)$ does not change sign at c, then $f(x)$ has no local maximum or minimum at c.

Example 24 Find the critical points of the function $f(x) = x^3 - 3x^2 - 9x + 6$, and classify each critical point as a local maximum, a local minimum, or neither.

Solution $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$

$x = 3$ and $x = -1$ are the critical points (where $f'(x) = 0$). Since $f'(x)$ is a polynomial function, it is differentiable everywhere. Thus, we have no points c such that $f'(c)$ is not defined.

x	$-\infty$	-1		3	$+\infty$
$f'(x)$	+	0	−	0	+
$f(x)$	↗		↘		↗
		(max)		(min)	

From the sign chart, f increases for $x < -1$ and decreases for $-1 < x < 3$. So, f has a local maximum at $x = -1$.

Similarly, f decreases for $-1 < x < 3$ and increases for $x > 3$. So, f has a local minimum at $x = 3$.

Example 25 Find the local extrema of the function $f(x) = x^{2/3} + 2$.

Solution $f'(x) = \dfrac{2}{3} \cdot x^{\frac{2}{3}-1} = \dfrac{2}{3 \cdot x^{\frac{1}{3}}}$

There is no root of $f'(x) = 0$.

Now we will look for the values of x such that $f'(x)$ is not defined but $f(x)$ is defined. We see that f is defined for all the values of x but f' is not defined at $x = 0$. So, 0 is a critical point.

x	$-\infty$	0	$+\infty$
$f'(x)$	−	0	+
$f(x)$	↘		↗
		(min)	

Thus, the first derivative test tells us that $x = 0$ is a local minimum of f.

Example **26** Find the local extrema of the function $f(x) = |x - 1|$.

Solution If $x > 1$, then $x - 1 > 0$. So, $f(x) = x - 1$.

If $x < 1$, then $x - 1 < 0$. So, $f(x) = 1 - x$.

$$f'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

Since $f'(1^-) \neq f'(1^+)$, $f'(x)$ is not defined at $x = 1$.

So, $x = 1$ is a critical point. Furthermore, $f'(x)$ is not equal to zero anywhere.

For $x > 1$, we have $f'(x) = 1 > 0$. So, $f(x)$ is increasing on this interval.

For $x < 1$, we have $f'(x) = -1 < 0$. So, $f(x)$ is decreasing on this interval.

x	$-\infty$		1		$+\infty$
$f'(x)$		$-$	o	$+$	
$f(x)$		$\searrow$	(min)	$\nearrow$	

Thus, f has a local minimum at $x = 1$.

Example **27** Find the local extrema of the function $f(x) = 5x^3 + 4x$.

Solution $f'(x) = 15x^2 + 4$ is always positive. There is no real solution of $f'(x) = 0$.

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$		$\nearrow$	

f is increasing for all the values of x.

Since $f(x)$ is a polynomial function, $f(x)$ is continuous and differentiable everywhere.

Thus, $f(x)$ has no local extrema.

Check Yourself 6

Find the local extrema of the following functions.

1. $f(x) = 2x^2 - 2x + 5$ 2. $f(x) = 1 - x^4$ 3. $f(x) = \dfrac{x^2+1}{x}$ 4. $f(x) = |x^2 - x|$

Answers

1. min.: $x = \dfrac{1}{2}$ 2. max.: $x = 0$ 3. max.: $x = -1$, min.: $x = 1$ 4. max.: $x = \dfrac{1}{2}$, min.: $x = 0$, $x = 1$

Example 28 If the function $f(x) = x^3 + ax^2 + 15x + b$ has a local maximum at the point (1, 10), then find a and b.

Solution We know that an extremum of a function must occur at a point where $f'(x) = 0$ or $f'(x)$ does not exist. Since f is a polynomial function, f is differentiable everywhere.

So, we have $f'(1) = 0$.

$f'(x) = 3x^2 + 2ax + 15$

$f'(1) = 0$

$3 \cdot 1^2 + 2a \cdot 1 + 15 = 0$

$a = -9$

Since the point (1, 10) is on the graph of f, we say that $f(1) = 10$.

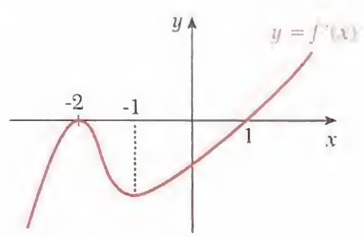
$f(1) = 10$

$1^3 + a \cdot 1^2 + 15 \cdot 1 + b = 10$

$a + b = -6$

$b = 3 \quad (\text{since } a = -9)$

Example 29 The graph of the derivative of the function $f(x)$ is shown in the figure. Find the intervals where $f(x)$ is increasing or decreasing and find the local extrema of f.



Solution We know that $f(x)$ is increasing when $f'(x) > 0$. $f'(x) > 0$ means that the graph of $f(x)$ must be above the x-axis. In the figure $f'(x) > 0$ for $x > 1$. So, $f(x)$ is increasing for $x > 1$. Similarly, $f(x)$ is decreasing when the graph of $f'(x)$ is below the x-axis. So, $f(x)$ is decreasing for $x < 1$.

x	$-\infty$		1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$		↘	(min)	↗	

From the chart, f has a local minimum at $x = 1$.

Example 30 For what values of m does the function $f(x) = x^3 - (m-1)x^2 + 3x - 2$ have no local extrema?

Solution Since f has no extrema, there must be no root of $f'(x) = 0$.

$f'(x) = 3x^2 - 2(m-1)x + 3$

The equation $3x^2 - 2(m-1)x + 3 = 0$ must have no root.

We need $\Delta < 0$:

$[-2(m-1)]^2 - 4 \cdot 3 \cdot 3 < 0$

$4(m^2 - 2m + 1) - 36 < 0$

$4(m^2 - 2m - 8) < 0$

$4(m-4)(m+2) < 0$

Let us construct a chart to solve the above inequality:

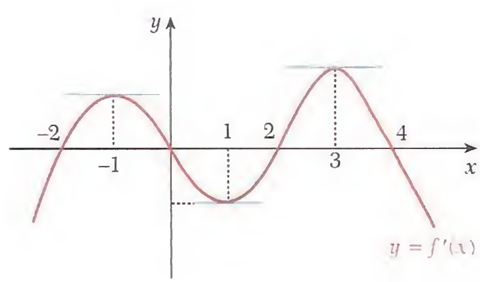
	-2		4	
$+$	o	$-$	o	$+$

So $-2 < m < 4$.

> A quadratic equation has
> 1. two solutions when $\Delta > 0$.
> 2. one solution when $\Delta = 0$.
> 3. no solution when $\Delta < 0$.

Check Yourself 7

1. Find the local minimum value of $y = e^{2x} - 4e^x - 6x$.
2. The function $f(x) = x^3 - 3x^2 - 9x + a$ has a local maximum value of 10. Find a.
3. The graph of the derivative of the function $f(x)$ is given. Find the local extrema of the function f.

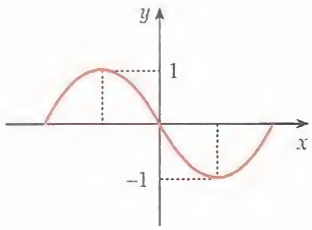


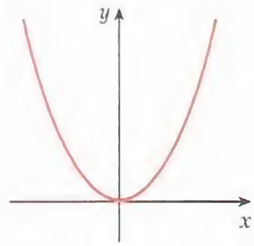
Answers

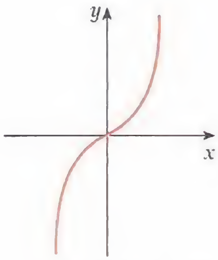
1. $-3 - 6\ln 3$ 2. 5 3. max.: $x = 0$, $x = 4$, min.: $x = -2$, $x = 2$

4. Finding the Absolute Extrema

In most types of maximum-minimum problems, we are more interested in the absolute extrema rather than the local extrema. Recall that the absolute extrema of a function are the largest and the smallest values of that function in its whole domain. The following figures show the graphs of several functions and give the maximum and minimum values of the functions if they exist.







We have seen that some functions have absolute extrema, whereas other do not. In what conditions does a function have both the absolute maximum and the absolute minimum? The following theorem answers this question.

Theorem

If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

The above theorem guarantees the existence of the absolute extrema of a continuous function on a closed interval $[a, b]$. Moreover, we know that each absolute extremum can occur either at a critical point in the interior of $[a, b]$ or at an endpoint of the interval. The following steps give a useful method for finding the absolute extrema of a continuous function on $[a, b]$.

CLOSED INTERVAL METHOD

1. Find the critical points of f on the interval $[a, b]$.
2. Evaluate $f(x)$ at each critical point.
3. Evaluate $f(a)$ and $f(b)$.
4. The largest of the values of f found in Steps 2 and 3 is the absolute maximum, the smallest of these values is the absolute minimum.

Example **31** Find the absolute extrema of the function $f(x) = x^2 - 4x + 3$ on $[0, 3]$.

Solution Let us apply the Closed Interval Method step by step:

1st Step: To find the critical points of f, we must solve $f'(x) = 0$ and also find where $f'(x)$ does not exist.

$f'(x) = 2x - 4 = 0$ gives $x = 2$.

The domain of the function is $[0, 3]$. So, $x = 2$ is in the domain.

And there is no point where $f'(x)$ is not defined.

Thus, the only critical point on $[0, 3]$ is $x = 2$.

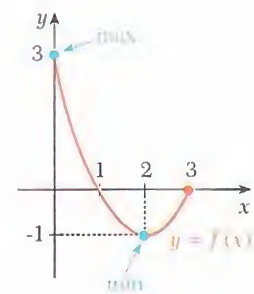
2nd Step: $f(2) = 2^2 - 4 \cdot 2 + 3 = -1$

3rd Step: $f(a) = f(0) = 0^2 - 4 \cdot 0 + 3 = 3$

$f(b) = f(3) = 3^2 - 4 \cdot 3 + 3 = 0$

4th Step: $f_{\max}[0, 3] = 3, \quad f_{\min}[0, 3] = -1.$

The graph of f confirms our results.



Example **32** Find the maximum and minimum values of the function $f(x) = 2x^3 + 12x^2 + 18x + 6$ on the closed interval $[-2, 0]$.

Solution $f'(x) = 6x^2 + 24x + 18 = 6(x + 1)(x + 3)$

$f'(x) = 0$ when $x = -1$ and $x = -3$.

But $x = -3$ is outside the interval $[-2, 0]$. So, we do not take it.

The only critical point is $x = -1$.

Additionally, we should consider the endpoints of the interval $[-2, 0]$.

Now, we evaluate $f(x)$ at $x = -1, -2$, and 0:

$f(-1) = -2$

$f(-2) = 2$

$f(0) = 6$

$f_{\max}[-2, 0] = 6, \; f_{\min}[-2, 0] = -2.$

Notation

$f_{\max}[a, b]$ denotes the maximum value of the function f on the interval $[a, b]$.

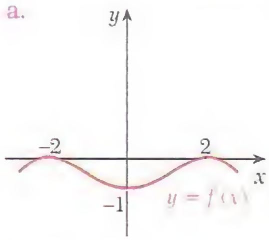
$f_{\min}[a, b]$ denotes the minimum value of the function f on the interval $[a, b]$.

EXERCISES 2.2

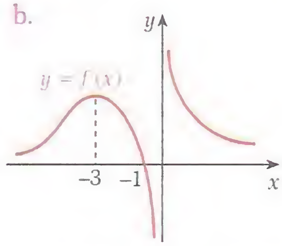
A. Intervals of Increase and Decrease

1. You are given the graphs of two functions. Determine where the functions are increasing and where they are decreasing.

a.



b.



2. Find the intervals where each of the following functions is increasing or decreasing.
   a. $f(x) = 3 - 8x$
   b. $f(x) = x^2 + 1$
   c. $f(x) = -x^2 + 4x + 3$
   d. $f(x) = x^3 + 6x$
   e. $f(x) = \dfrac{x^3}{3} - 2x^2 + 2$
   f. $f(x) = 3x^4 + 4x^3 - 12x^2$
   g. $f(x) = \dfrac{1}{2-x}$
   h. $f(x) = x^{2/3}$
   i. $f(x) = \dfrac{5-x}{x^2}$
   j. $f(x) = \sqrt[3]{x-1}$
   k. $f(x) = 2^{\frac{1}{x-5}}$
   l. $f(x) = e^{x^2 - 4x + 3}$
   m. $f(x) = \dfrac{\ln x}{x^2}$
   n. $f(x) = \sin\left(x + \dfrac{\pi}{3}\right)$
   o. $f(x) = \dfrac{0.5^{2x}}{2\ln 0.5} - \dfrac{4 \cdot 0.5^x}{\ln 0.5} + 3x - 2$

3. Find the intervals where the function $f(x) = \sin x + \cos x$ is decreasing on $[0, 2\pi]$.

4. Show that the function $f(x) = \text{Arctan}x$ is increasing for all the values of x.

5. For what values of m is the function $f(x) = -\dfrac{1}{3}x^3 + mx^2 - 4x + 1$ decreasing for all real numbers?

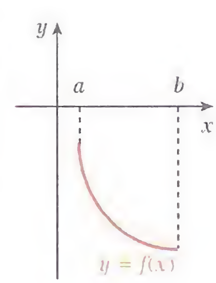
6. The function $f(x) = ax^3 - (a-2)x^2 + \dfrac{1}{3}x$ is always increasing for all the values of x. Find a.

7. Find the values of a, so that $f(x) = \dfrac{x^2 - ax}{x^2 - 4x + 3}$ is always decreasing in its domain.

8. Find a, so that the function $f(x) = x^3 - 3x^2 + 3ax + 15$ is increasing on $(-\infty, -2)$ and $(4, \infty)$, and decreasing on $(-2, 4)$.

9. Let f be an increasing function on $(0, \infty)$. State whether each of the following functions are increasing or decreasing on the same interval.
   a. $-f(x)$
   b. $x + f(x)$
   c. $\dfrac{1}{f(x)}$
   d. $f(x^2)$

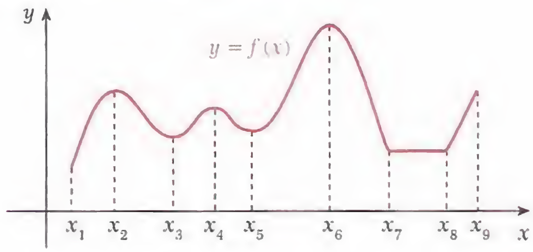
10. The graph of the function f is given on the closed interval $[a, b]$. State whether each of the following functions are increasing or decreasing on $[a, b]$.



   a. $x \cdot f(x)$
   b. $f^2(x) + x$
   c. $x^2 - f(x)$

B. Maximum and Minimum Values

11. For each of the points from x_1 to x_9, state whether f has a local maximum or minimum, and an absolute maximum or minimum.

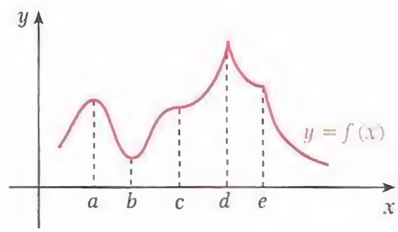


12. Find the critical points of the following functions.

a. $f(x) = x^3 + x$ b. $f(x) = x^2e^x$

c. $f(x) = 2x - \dfrac{1}{x^2}$ d. $f(x) = |x + 1|$

13.



In the figure above the graph of a function f with the critical points at a, b, c, d, and e are shown.

a. State why these points are critical.

b. Classify each of them as a local maximum, a local minimum, or neither.

14. Find the local extrema of the following functions.

a. $f(x) = 8x + x^2$

b. $f(x) = -x^3 + 3x + 2$

c. $f(x) = \dfrac{x^3}{3} + 2x^2 + 4x + 5$

d. $f(x) = (x - 1)^2(x + 3)^2$

e. $f(x) = \dfrac{x}{x+1}$

f. $f(x) = x^2 - \dfrac{16}{x}$

g. $f(x) = |4 - x^2|$

✪ h. $f(x) = \left(\dfrac{1}{2}\right)^{3-6x-x^2}$

✪ i. $f(x) = x^2 - 3x + x\ln x$

✪✪ j. $f(x) = \sin^2 x + \sin x$

15. The function $f(x) = x^3 - 9x^2 + 15x + 7$ is given. Find the sum of the x-coordinates of its local extrema.

16. The graph of $y = ax^2 + bx$ has an extremum at $(1, -2)$. Find the values of a and b.

17. Find k, if $f(x) = x^3 - 2x^2 - 7x + k$ has a local maximum value of 8.

18. Given that $f(x) = x^3 + ax^2 + bx + 1$ has a local maximum at $x = -1$ and a local minimum at $x = 2$, find a and b.

19. Find the local minimum value of the function $f(x) = x^2e^x - 3e^x$.

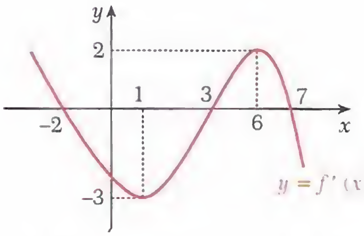
20. Find the value of m, if the curve $y = x^3 + 2mx^2 + 30$ is tangent to the line $y = -2$.

21. Find the relation between a and b, if the function $f(x) = ax^3 + bx + c$ has one local maximum and one local minimum.

22. For what values of m does the function

$$f(x) = \frac{mx^2+1}{x-1}$$ have no local extrema?

23.



The graph of the derivative of a function f is given. Find the local extrema of f.

24. Find the absolute extrema of each function on the given interval.

a. $f(x) = 2x^2 - 4x + 3,$ $[0, 2]$

b. $f(x) = -x^2 + 2x - 1,$ $[-2, 2]$

c. $f(x) = x^3 - 6x,$ $[1, 4]$

d. $f(x) = 2x^3 - 15x^2 + 24x + 19,$ $[0, 2]$

e. $f(x) = x^2 - 4\sqrt{x},$ $[0, 3]$

f. $f(x) = x^5 - 5x^4 + 1,$ $[0, 5]$

g. $f(x) = 9x^2 - x^4,$ $[-3, 3]$

h. $f(x) = \sqrt{9 - x^2},$ $[-1, 2]$

i. $f(x) = x - \dfrac{1}{x},$ $[1, 3]$

j. $f(x) = 3x^{2/3},$ $[-1, 1]$

k. $f(x) = \dfrac{x-1}{x+1},$ $[0, 4]$

l. $f(x) = |x^2 - 3x|,$ $[1, 4]$

m. $f(x) = \dfrac{1}{3}\log_2^3 x - 3\log_2^2 x + 8\log_2 x + 1,$ $[1, 8]$

n. $f(x) = 9\sin x - \sin 3x + 3,$ $[-\pi, 0]$

25. Find the sum of the smallest value and the greatest value of $f(x) = x^2 - 4x + 8$ on $[-2, 3]$.

26. If the point $(1, 4)$ is the highest point of the graph of $f(x) = ax^2 + 2x + b$, find $a + b$.

27. Let $f(x) = ax^3 - bx$. Find a and b, if $f(2) = 4$ is the maximum value of f on $[0 , 4]$.

28. Find the maximum value of $f(x) = \sin x + \cos x$.

Mixed Problems

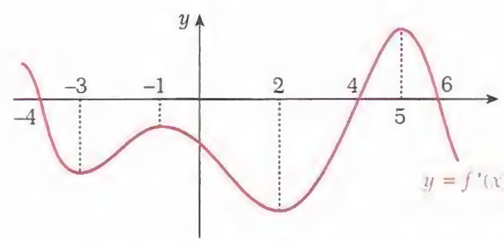
29. Given a parametric function $y = f(x)$ with

$$y = 2t^2 + 4t + 5$$
$$x = t^3 + t.$$

a. Find the intervals of increase and decrease of f.

b. Find the local extrema of f.

30.



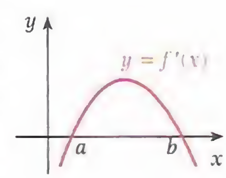
The graph of the derivative of a function f is given.

a. Find the intervals of increase and decrease of f.

b. Find the local extrema of f.

31. At what point does the tangent to the curve $y = \dfrac{x^3}{3} - 2x^2 + x - 5$ have the smallest slope?

32. In the figure, graph of $f'(x)$ is given. Given that the equation $f(x) = 0$ has only one root and that root is positive, plot a rough graph of $f(x)$.



33. Find the range of the function

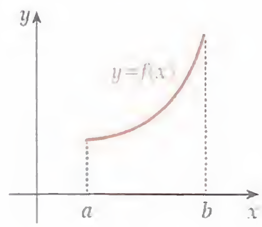
$$f(x) = \begin{cases} 3x^4 - 4x^3 - 24x^2 + 48x, & x \geq 0.5 \\ 8x^3 + 12x^2 + 2, & x < 0.5 \end{cases}.$$

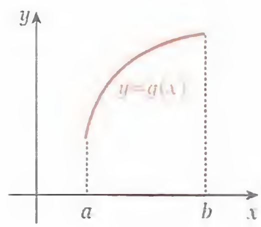
34. For which values of a does the interval $\left[0, \dfrac{1}{3}\right]$ completely include the range of the function $f(x) = \dfrac{1}{3x^4 - 8ax^3 + 12a^2x^2 + a}$?

3 APPLICATIONS OF THE SECOND DERIVATIVES

A. CONCAVITY

In this section we discuss the concept of concavity. As illustrated in the following figures, two increasing graphs on an interval may have different shapes. This depends on how the graphs bend or turn. As we scan the graphs from left to right, we see that the graph of f turns to the left (upward), while the graph of g turns to the right (downward). We say that the function f is concave up on the interval (a, b) and the function g is concave down on the interval (a, b). We now define concavity geometrically.





Definition

concavity

A function f is **concave up** on an interval I if the graph of f lies above all of its tangent lines on the interval I.

Similarly, f is **concave down** on I if the graph of f lies below all of its tangent lines on I.

The graphs above illustrates the definition of concavity. Now, we shall see that the second derivative f'' tells us where f is concave up and where f is concave down. If f is concave up on (a, b), then the slopes of the tangent lines increase from left to right as shown in the left figure above. This means that the first derivative f' is increasing on (a, b). We know that if f' is an increasing function, then its derivative f'' must be positive on (a, b). In a similar way, it can be shown that if f is concave down on (a, b), then $f''(x) < 0$ on (a, b). These observations suggest the following theorem.

$f' > 0 \Leftrightarrow f$ is increasing
$f' < 0 \Leftrightarrow f$ is decreasing

Theorem

Let the function f be twice differentiable on the interval I.

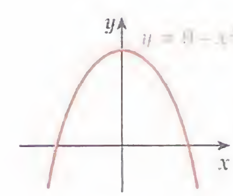
1. If $f''(x) > 0$ for all the values of x on the interval I, then f is concave up on I.
2. If $f''(x) < 0$ for all the values of x on the interval I, then f is concave down on I.

EXAMPLE 35 Determine where the following functions are concave up and where they are concave down.

a. $f(x) = 9 - x^2$ b. $f(x) = e^x$ c. $f(x) = x^3$

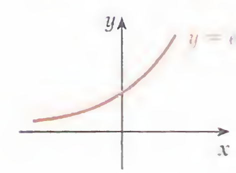
Solution

a. $f(x) = 9 - x^2$
$f'(x) = -2x$ and $f''(x) = -2$
Since $f''(x) < 0$ for all the values of x, f is concave down everywhere.



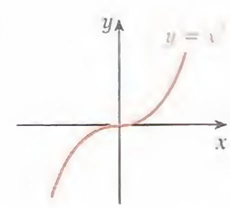
As a conclusion of the above theorem, we must examine the sign of the second derivative.

b. $f(x) = e^x$
$f'(x) = e^x$ and $f''(x) = e^x$
As shown on the right, f is concave up on $(-\infty, \infty)$ because $f''(x) > 0$ for all the values of x.



c. $f(x) = x^3$
$f'(x) = 3x^2$ and $f''(x) = 6x$
Setting $f''(x) = 0$ gives $x = 0$.

x	$-\infty$	0	$+\infty$
$f''(x)$	$-$	0	$+$
$f(x)$	concave down		concave up

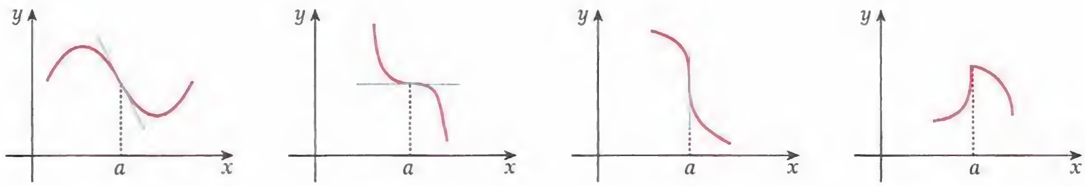


From the sign chart, $f''(x)$ changes sign from negative to positive at the point $x = 0$. Observe that the point $(0, 0)$ on the graph of $f(x) = x^3$ is where f changes from concave down to concave up. We call it the **inflection point** of f.

Definition

Inflection point

An **inflection point** is a point where a graph changes its direction of concavity.



$f''(a) = 0$ $\quad$ $f''(a) = 0$ $\quad$ $f''(a)$ does not exist $\quad$ $f''(a)$ does not exist

Note

At each inflection point, either

1. $f''(a) = 0$ or
2. $f''(a)$ does not exist.

FINDING THE INFLECTION POINTS

To find the inflection points of a function, follow the steps.

1. Find the points where $f''(x) = 0$ and $f''(x)$ does not exist. These points are the possible inflection points of the function f.
2. Construct the sign chart of $f''(x)$. If the sign of $f''(x)$ changes across the point $x = a$, then $(a, f(a))$ is an inflection point of f.

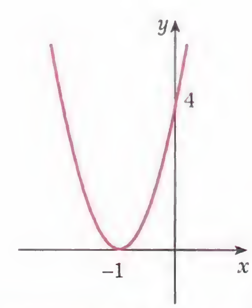
Example 36 Investigate $f(x) = (x + 1)^4$ for concavity and find the inflection points.

Solution $f'(x) = 4(x + 1)^3$

$f''(x) = 12(x + 1)^2$

$x = -1$ is a double root of $f''(x) = 0$.

x	$-\infty$	-1	$+\infty$
$f''(x)$	$+$	0	$+$
$f(x)$	concave up		concave up



the graph of $f(x) = (x + 1)^4$

From the sign chart, $f(x)$ is concave up for all the values of x.

Also, we have $f''(x) = 0$ when $x = -1$.

But $(-1, 0)$ is not the inflection point of f because f'' does not change sign across $x = -1$.

Example 37 Find the intervals of concavity and the inflection points for $f(x)=\dfrac{1}{2}x^4-4x^3+9x^2-7x+5$.

Solution $f'(x) = 2x^3 - 12x^2 + 18x - 7$

$f''(x) = 6x^2 - 24x + 18 = 6(x-3)(x-1)$

Because $f''(x)$ exists everywhere, the possible inflection points are the solutions of the equation $f''(x) = 0$; that is, $x = 1$ and $x = 3$.

x	$-\infty$		1		3		$+\infty$
$f''(x)$		+	0	−	0	+	
$f(x)$		concave up	(inf)	concave down	(inf)	concave up	

From the sign chart for f'', we see that f is concave up on $(-\infty, 1)$ and $(3, \infty)$ and concave down on $(1, 3)$. It would also be true to say that f is concave up on $(-\infty, 1]$ and $[3, \infty)$ and concave down on $[1, 3]$.

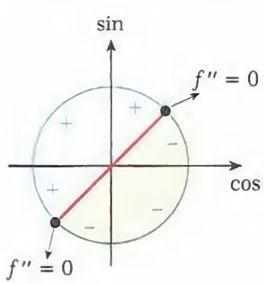
Also, observe that $f''(x)$ changes sign at $x = 1$ and $x = 3$. Therefore, the points $(1, f(1))$ and $(3, f(3))$ are the inflection points of f.

Example 38 Find the intervals of concavity and the inflection points for $f(x) = \cos x - \sin x$ on $[0, 2\pi]$.

Solution $f'(x) = -\sin x - \cos x$ and $f''(x) = -\cos x + \sin x$

Since f'' is differentiable on $[0, 2\pi]$, we must find the solutions of $f''(x) = 0$ on $[0, 2\pi]$.

$f''(x) = 0 \Rightarrow -\cos x + \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x=\dfrac{\pi}{4}$ and $x=\dfrac{5\pi}{4}$.



x	0		$\frac{\pi}{4}$		$\frac{5\pi}{4}$		2π
$f''(x)$		−	0	+	0	−	
$f(x)$		concave down	(inf)	concave up	(inf)	concave down	

The sign chart of f'' shows that f is concave down on $(0, \frac{\pi}{4})$ and $(\frac{5\pi}{4}, 2\pi)$ and concave up on $(\frac{\pi}{4}, \frac{5\pi}{4})$.

$$f(\frac{\pi}{4}) = \cos\frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \text{ and } f(\frac{5\pi}{4}) = \cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$$

So, $(\frac{\pi}{4}, 0)$ and $(\frac{5\pi}{4}, 0)$ are the inflection points of f.

Example 39 The function $f(x) = x^3 + ax^2 + bx + 3$ has an inflection point at (1, 3). Find the values of a and b.

Solution $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$

We know that the inflection point of f occurs at a point where $f''(x) = 0$ or $f''(x)$ does not exist.

Since (1, 3) is an inflection point, $f''(1) = 0$.

Also we have $f(1) = 3$ because the point (1, 3) is on the graph of f.

$f''(1) = 0 \Rightarrow 6 \cdot 1 + 2a = 0 \Rightarrow a = -3$

$f(1) = 3 \Rightarrow 1^3 + a \cdot 1^2 + b \cdot 1 + 3 = 3 \Rightarrow a + b = 0 \Rightarrow b = 3$

Example 40 Find the equation of the tangent line to the curve $f(x) = x^2 + \dfrac{1}{x}$ at its inflection point.

Solution We first need to find the inflection point of f.

$f'(x) = 2x - \dfrac{1}{x^2}$ and $f''(x) = 2 + \dfrac{2}{x^3}$

Setting $f''(x) = 0$ gives $x = -1$ and (–1, 0) is the inflection point.

f'' does not exist when $x = 0$ but this point is not the inflection point of f. (Why?)

Now we can find the equation of the tangent line at the point (–1, 0). The slope of the equation is

$$m = f'(-1) = 2 \cdot (-1) - \frac{1}{(-1)^2} = -2 - 1 = -3.$$

Using the point-slope form of a line,

$y - y_1 = m \cdot (x - x_1)$

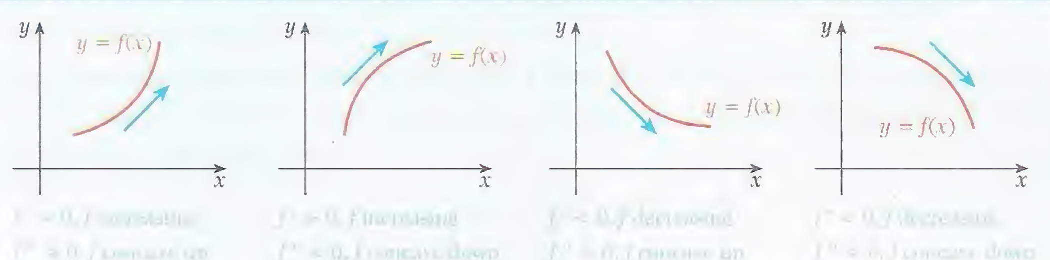
$y - 0 = -3 \cdot (x + 1)$ or $y = -3x - 3$.

Note

In the beginning of this section we have seen that an increasing graph can be either concave up or concave down. This shows that the increase and decrease of a function is independent of the concavity of the function.

Remember that the sign of the first derivative determines where f is increasing and decreasing, whereas the sign of the second derivative determines where f is concave up and concave down.

RELATIONSHIP BETWEEN A GRAPH AND ITS DERIVATIVES



Check Yourself 9

1. Find the intervals of concavity and the inflection points for each function.

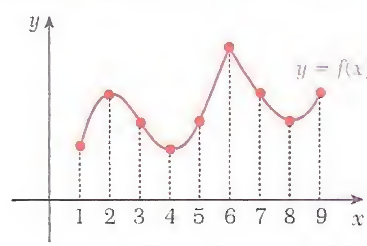
   a. $f(x) = x^3 - 2x^2 - 7x + 3$ b. $f(x) = \dfrac{x+1}{x-1}$ c. $h(x) = x + e^x$

2. The function $f(x) = x^3 + ax^2 + bx + 2$ has an inflection point at $(1, -1)$. Find a and b.

3. The graph of a function $y = f(x)$ is shown in the figure.

   a. Find the intervals of increase and decrease of f.

   b. Find the intervals of concavity of f.



Answers

1. a. concave up: $(\frac{2}{3}, \infty)$, concave down: $(-\infty, \frac{2}{3})$, inflection point $x = \frac{2}{3}$.

   b. concave up: $(1, \infty)$, concave down: $(-\infty, 1)$, no inflection point.

   c. concave up: $(-\infty, \infty)$, no inflection point.

2. $a = -3$, $b = -1$ 3. a. increasing: $(1, 2)$, $(4, 6)$, $(8, 9)$ b. concave up: $(3, 6)$, $(6, 9)$
   decreasing: $(2, 4)$, $(6, 8)$ concave down: $(1, 3)$

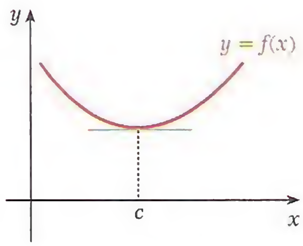
B. THE SECOND DERIVATIVE TEST

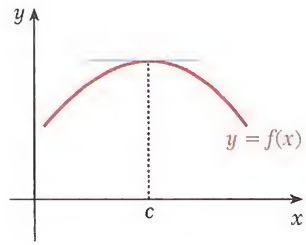
We have seen that the first derivative test helps us classify the critical points of a function f. Here we learn an alternative test for determining whether a critical point of f is a local maximum or a local minimum.

THE SECOND DERIVATIVE TEST

Let f be twice differentiable on an interval I and c be a critical point of f in I such that $f'(c) = 0$.

1. If $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f''(c) < 0$, then f has a local maximum at $x = c$.





The graphs above illustrates the second derivative test. We know that f is concave up near c if $f''(c) > 0$. This means that the graph of f lies above its horizontal tangent at c and so f has a local minimum at c.

Example 41 Apply the second derivative test to find the local extrema of the function $f(x) = x^3 - 3x^2 - 9x + 6$.

Solution $f'(x) = 3x^2 - 6x - 9$ and $f''(x) = 6x - 6$.

So, $f'(x) = 0$ gives $x = -1$ and $x = 3$, the critical points of f.

To apply the second derivative test, we compute f'' at these points.

$f''(-1) = -12$ and $f''(3) = 12$

Since $f''(-1) < 0$, the second derivative test implies that $f(-1) = 11$ is a local maximum value of f. And since $f''(3) > 0$, it follows that $f(3) = -21$ is a local minimum value.

Remember that we had found the same results by using the first derivative test in Example 24.

Note

The second derivative test can be used only when f'' exists. Moreover, this test fails when $f''(c) = 0$. In other words, if $f'(c) = 0 = f''(c)$, then there might be a local maximum, a local minimum, or neither at the point $x = c$. In such cases we must use the first derivative test.

Example 42 Find the local extrema of the function $f(x) = 3x^5 - 5x^3 + 3$.

Solution $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1) = 0$

So, the critical points are $x = -1$, $x = 0$, and $x = 1$.

The second derivative is $f''(x) = 60x^3 - 30x$.

When we compute $f''(x)$ at each critical point, we find that

$f''(-1) = -30 < 0, f''(0) = 0, f''(1) = 30 > 0$.

The second derivative test tells us that f has a local maximum at $x = -1$, a local minimum at $x = 1$. Since $f''(0) = 0$, this test gives no information about the critical point 0. Let us apply the first derivative test.

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		+	0	−	0	−	0	+	
$f(x)$		↗		↘		↘		↗	

We see that f' does not change sign at $x = 0$. So, f does not have a local maximum or minimum.

Check Yourself 10

Apply the second derivative test to find the local extrema of each function.

1. $f(x) = 4x^3 + 9x^2 - 12x + 7$
2. $f(x) = 8x^5 - 5x^4 - 20x^3$
3. $f(x) = x + \dfrac{4}{x}$

Answers

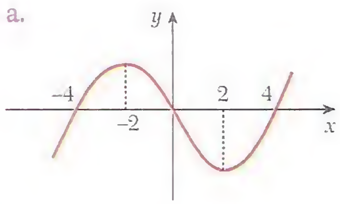
1. max.: $x = -2$ min.: $x = \dfrac{1}{2}$
2. max.: $x = -1$ min.: $x = \dfrac{3}{2}$
3. max.: $x = -2$ min.: $x = 2$

EXERCISES 2.3

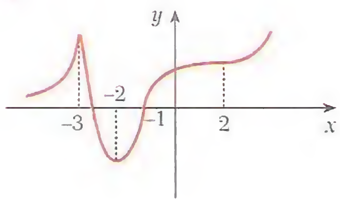
A. Concavity

1. The graphs of two functions are given. Find the intervals where the second derivative of the function is positive or negative.

a.



b.



2. Find the intervals of concavity and the inflection points for each function.

a. $f(x) = x^2 - 5x + 6$

b. $f(x) = -2x^2 + 7x$

c. $f(x) = x^3 + x^2$

d. $f(x) = x^3 - 3x^2 + 5x - 7$

e. $f(x) = 3x^4 - 16x^3 + 30x^2 + 4$

f. $f(x) = \frac{1}{4}x^4 - 6x^2 + 4x - 7$

g. $f(x) = 3x^5 - 10x^3 + 5x$

h. $f(x) = \frac{1}{x^2}$

i. $f(x) = \sqrt{x+1}$

j. $f(x) = \sqrt[3]{5-x}$

✪ k. $f(x) = \frac{e^x - e^{-x}}{2}$

✪ l. $f(x) = -\text{Arcsin}(x-2)$

✪✪ m. $f(x) = \frac{\sin x}{1 - \cos x}$

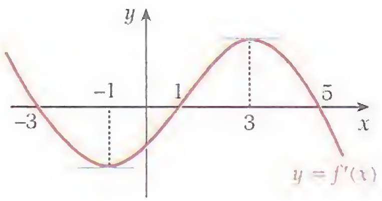
3. Find a and b, if $f(x) = x^4 - 4x^3 + ax^2 + b$ has an inflection point at $(1, 3)$.

4. The function $f(x) = x^4 + kx^2 + 7x - 7$ has an inflection point at $x = 1$. Find the coordinates of the other inflection point.

5. ✪ Find the coordinates of the inflection points of the function $f(x) = \frac{e^x}{x^2+1}$ on the interval $(0, \infty)$.

6. Find the coordinates of the inflection points of the function $f(x) = e^x \sin x$ on the interval $[0, \pi]$.

7. The graph of the derivative of a function f is shown in the figure. Find the intervals of concavity and the inflection points of f.



B. The Second Derivative Test

8. Determine whether f has a local maximum or minimum at the given value of x, using the second derivative test.

a. $f(x) = \frac{1}{3}x^3 - 3x^2 - 7x + 5, \quad x = 7$

b. $f(x) = x^4 - 3x^2 + 2, \quad x = 0$

c. $f(x) = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}, \quad x = -1$

d. $f(x) = x^2 + \frac{2}{x}, \quad x = 1$

9. Find the local extrema of the following functions, using either the first or second derivative test.

a. $f(x) = x^3 + 6x^2 + 9x + 1$

b. $f(x) = x^4 + 4x^3 + 2x^2 + 1$

c. $f(x) = \frac{x^2}{x-2}$

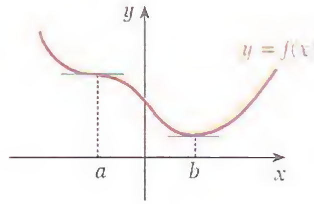
d. $f(x) = (x+3)^{\frac{2}{3}}$

e. $f(x) = \frac{1}{1+x^2}$

f. $f(x) = x + \sin x$

Mixed Problems

10. Given the graph of $f(x)$, plot a rough graph of $f'(x)$.



11. Show that a cubic function has exactly one inflection point.

12. Show that a polynomial function of degree 4 has either no inflection point or exactly two inflection points.

13. Find the equation of the tangent line to the curve $y = \frac{1}{3}x^3 - 2x^2 + 3x + \frac{1}{3}$ at its inflection point.

14. Find a, b, and c so that $f(x) = ax^3 - 3x^2 + bx + c$ has an inflection point at the point $(-1, 1)$ and a local extremum at $x = -2$.

15. The tangent line to the curve $f(x) = x^3 + 3x^2 + cx + 1$ at the inflection point of f is perpendicular to the line $y = x + 4$. Find c.

16. Given that $g(x) = e^{-x} \cdot f(x)$ where f is a differentiable function for all real numbers and the function $g(x)$ has an inflection point at $x = a$, find an expression for $f''(a)$ in terms of $f'(a)$ and $f(a)$.

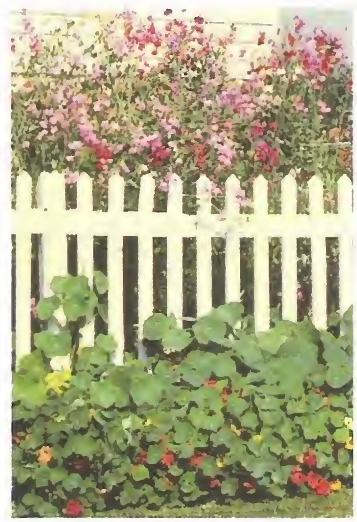
4 OPTIMIZATION PROBLEMS

In this section we solve applied maximum-minimum problems in which the function is not given directly. When we face such a problem, we are required to first find the appropriate function to be maximized or minimized. The following steps will be helpful for solving these problems.

1. Determine the quantity to be maximized or minimized and label it with a letter (say M for now).
2. Assign letters for other quantities, possibly with the help of a figure.
3. Express M in terms of some of the other variables.
4. Use the data in the problem to write M as a function of one variable x, say $M = M(x)$.
5. Find the domain of the function $M(x)$.
6. Find the maximum (or minimum) value of $M(x)$ with the help of the first derivative.

Such problems where we look for the "best" value are called **optimization problems.**

Example 43 A man has 40 m of fencing that he plans to use to enclose a rectangular garden plot. Find the dimensions of the plot that will maximize the area.



Solution We want to maximize the area A of the rectangular plot.

Let x and y represent the length and width of the rectangle. Then, since there is 40 m of fencing,

$2x + 2y = 40$ or $x + y = 20$.

Then we express A in terms of x and y: $A = x \cdot y$

Expressing A as a function of just one variable,

we get $A(x) = x \cdot (20 - x) = 20x - x^2$ (since $y = 20 - x$).

Since the dimensions will be positive,

$x > 0$ and $y = 20 - x > 0$ or $0 < x < 20$.

Given a rectangle with sidelength a and b

Perimeter $= 2(a + b)$

Area $= a \cdot b$

The derivative is $A'(x) = 20 - 2x$. So, the only critical point is $x = 10$. To investigate this critical point, we calculate the second derivative. Since $A''(x) = -2 < 0$, the second derivative test implies that A has a local maximum $x = 10$.

We can verify that this local maximum is the absolute maximum by showing that the graph of A is concave down everywhere. Since $A''(x) < 0$ for all the values of x in $(0, 20)$, maximum value of A occurs at $x = 10$. The corresponding value of y is $y = 20 - x = 20 - 10 = 10$.

Thus, the garden would be of maximum area (100 m^2) if it was in the form of a square with sides 10 m.

Note

Suppose that f has only one critical point c in the interval I. If $f''(x)$ has the same sign at all points of I, then $f(c)$ is an absolute extremum of f on I. This absolute interpretation of the second derivative test is useful in optimization problems.

Example 44 Find two positive numbers x and y such that their sum is 15 and $x^2 + 5y$ is as small as possible.

Solution We have $x + y = 15$ and we want to minimize $M = x^2 + 5y$. Expressing M as a function of just one variable we get $M(x) = x^2 + 5(15 - x) = x^2 - 5x + 75$ (since $y = 15 - x$).

Since both numbers are positive, $x > 0$ and $y = 9 - x > 0$ or $0 < x < 9$.

The derivative is $M'(x) = 2x - 5$. So, the critical point is $x = \frac{5}{2}$.

Since $M''(x) = 2 > 0$ for all the values of x in $(0, 9)$, the second derivative test implies that $x = \frac{5}{2}$ is the minimum value of $M(x)$.

Therefore, M gets its minimum value when $x = \frac{5}{2}$, and $y = 15 - x = 15 - \frac{5}{2} = \frac{25}{2}$.

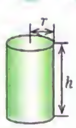
Example 45 A manufacturer has an order to make cylindrical cans with a volume of 500 cm^3. Find the radius of the cans that will minimize the cost of the metal in their production.

Solution In order to minimize the cost of the metal, we minimize the total surface area of the cylinder.

The volume of the can is $V = \pi r^2 h = 500$. So, $h = \frac{500}{\pi r^2}$.

Hence the surface area of the can as a function of r is

$$A(r) = 2\pi r^2 + 2\pi r \cdot \left(\frac{500}{\pi r^2}\right) = 2\pi r^2 + \frac{1000}{r}, \quad r > 0.$$



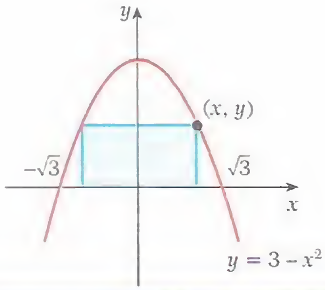
The surface area of a cylinder is $2\pi r^2 + 2\pi rh$ and the volume is $\pi r^2 h$ where r is the radius and h the height.

The derivative of $A(r)$ is $A'(r) = 4\pi r - \frac{1000}{r^2} = \frac{4(\pi r^3 - 250)}{r^2}$.

Thus, the only critical point is $r = \sqrt[3]{\frac{250}{\pi}}$.

Since $A''(r) = 4\pi + \frac{2000}{r^3} > 0$ for $r = \sqrt[3]{\frac{250}{\pi}}$, the second derivative test implies that A gets its minimum value when $r = \sqrt[3]{\frac{250}{\pi}}$.

Example 46 Find the area of the largest rectangle that has two vertices on the x - axis and another two above the x - axis on the parabola $y = 3 - x^2$.



Solution Let (x, y) be the vertex of the rectangle in the first quadrant. Then the rectangle has sides with the lengths $2x$ and y. So, its area is $A = 2xy$.

Using the fact that (x, y) lies on the parabola $y = 3 - x^2$, the expression to be maximized is $A(x) = 2 \cdot x \cdot (3 - x^2) = 6x - 2x^3$.

The domain of this function is $(0, \sqrt{3})$.

Its derivative is $A'(x) = 6 - 6x^2$.

So, the critical points are -1 and 1.

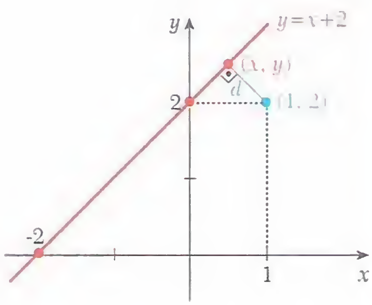
Only the positive value $x = 1$ lies in the interval $(0, \sqrt{3})$. Since this is the only critical value in the interval, we can apply the second derivative test.

The second derivative is $A''(x) = -12x$ and in particular $A''(x) < 0$ for all the values of x in $(0, \sqrt{3})$. So, the maximum value of $A(x)$ in this interval is $A(1) = 4$. Therefore, the area of the largest rectangle is 4.

Example 47 Find the point on the line $y = x + 2$ that is the closest to the point $(1, 2)$.

Solution Let (x, y) be a point on $y = x + 2$ such that the distance d between (x, y) and $(1, 2)$ is a minimum.

> Distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.



We have $d = \sqrt{(x-1)^2 + (y-2)^2}$.

If the point (x, y) is on the line, then $y = x + 2$. To rewrite d in terms of the single variable x, substitute $y = x + 2$.

After substitution, $d = \sqrt{(x-1)^2 + x^2} = \sqrt{2x^2 - 2x + 1}$.

It is clear that the minimum of d occurs at the same point as the minimum of d^2. So, we minimize d^2 to simplify calculations by letting $M = d^2$.

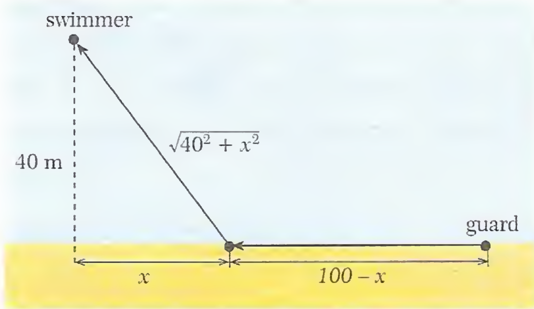
$M(x) = 2x^2 - 2x + 1$ has derivative $M'(x) = 4x - 2$.

So, $M'(x) = 0$ when $x = \frac{1}{2}$. Since $M''(x) = 4 > 0$, the minimum value occurs at $x = \frac{1}{2}$.

Since $y = \frac{1}{2} + 2 = \frac{5}{2}$, the point $(\frac{1}{2}, \frac{5}{2})$ is the closest point to the point $(1, 2)$.

Example 48 The swimmer is 40 m from the shoreline. The lifeguard is 100 m from the point on the shore that is directly opposite the swimmer. The guard can run at a speed of 5 m/s and swim at a speed of 3 m/s. What path should the guard follow to get to the swimmer in the least time?

Solution Let x be the distance denoted in the given diagram.



Recall that if travel is at a constant rate of speed, then

(distance traveled) = (rate of travel) · (time elapsed).

In short, $D = R \cdot T$ or $T = \dfrac{D}{R}$.

The total time elapsed is

$$T(x) = (\text{swim time}) + (\text{ run time}) = \frac{\text{swim distance}}{\text{swim rate}} + \frac{\text{run distance}}{\text{run rate}} = \frac{\sqrt{40^2 + x^2}}{3} + \frac{100 - x}{5}.$$

We wish to minimize the total time elapsed. So, we differentiate this equation to get

$$T'(x) = \frac{1}{3} \cdot \frac{1}{2} \cdot (40^2 + x^2)^{-1/2} \cdot (40^2 + x^2)' - \frac{1}{5} = \frac{x}{3\sqrt{40^2 + x^2}} - \frac{1}{5}.$$

$T'(x) = 0$ gives $5x = \sqrt{40^2 + x^2} \Rightarrow 25x^2 = 9(40^2 + x^2) \Rightarrow 16x^2 = 9 \cdot 40^2 \Rightarrow x = \pm 30$.

But $x \neq -30$ since x measures a distance.

It is left to the student to verify that $T''(30) > 0$ which means $x = 30$ corresponds to a path of minimum time.

Check Yourself 11

1. Find two positive numbers x and y such that their sum is 9 and x^2y is as large as possible.
2. A rectangle has area of 144 m². What dimensions will minimize its perimeter?
3. An open rectangular box with a square base is to be made from 300 cm² of material. Find the dimensions of the box with the maximum volume.
4. Find the minimum distance from the line $2x + 3y = 13$ to the origin.

Answers

1. $x = 6, y = 3$ 2. 12, 12 3. 5, 10 4. $\sqrt{13}$

REFRACTION OF LIGHT

How can we explain why a pencil appears to be broken when it is immersed partially into water or why objects under water appear to be nearer the surface than they really are to an observer looking down? It is an illusion caused by the **refraction** of light.

When light passes from one transparent medium to another(like air and water), it changes speed, and bends according to the law of refraction which states:

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

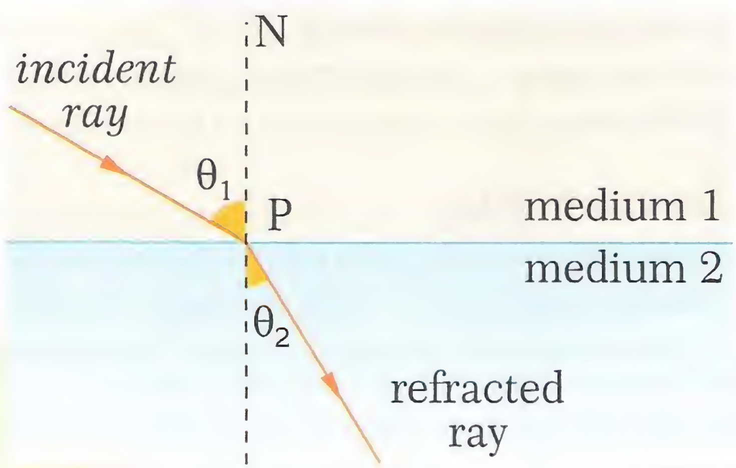
where,

v_1 is the speed of light in medium 1,

v_2 is the speed of light in medium 2,

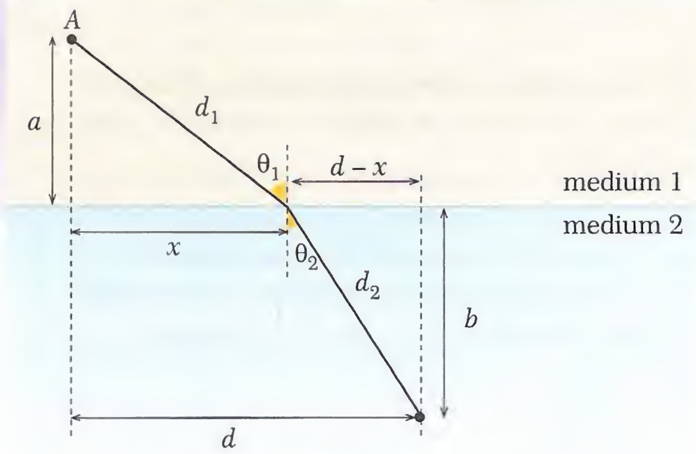
θ_1 is the angle between the incident ray and normal to the surface at the point P,

θ_2 is the angle between the refracted ray and the normal.



The experimental discovery of this relationship is usually credited to Willebrord Snell (1591-1627) and is therefore known as **Snell's law**.

Snell's law can be derived from a physics principle discovered by Pierre de Fermat, the seventeenth-century mathematician. Fermat's principle states that a ray of light travels the path of minimum time. The derivation of Snell's law from Fermat's principle represent an interesting application of the derivative.



Suppose a light ray is to travel from A to B, where A is in the medium 1 and B is in the medium 2. Using the geometry of the figure given above, we see that the time it takes the ray to travel from A to B is

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2} \qquad \left(\text{time} = \frac{\text{distance}}{\text{velocity}}\right),$$

$$t(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{(d-x)^2 + b^2}}{v_2}.$$

We obtain the least time, or the minimum value of t, by taking the derivative of t with respect to x and setting the derivative equal to zero.

$$t'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{d-x}{v_2\sqrt{(d-x)^2 + b^2}}$$

The function t is differentiable for all the values of x. So, the only critical values are the solutions to the equation $t'(x) = 0$. This equation gives the condition that

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{d-x}{v_2\sqrt{(d-x)^2 + b^2}}.$$

From the figure, we see that $\sin\theta_1 = \dfrac{x}{\sqrt{a^2 + x^2}}$ and $\sin\theta_2 = \dfrac{d-x}{\sqrt{(d-x)^2 + b^2}}$.

Using these equations, we obtain $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$, which is Snell's law.

EXERCISES 2.4

1. Find two positive numbers such that their sum is 30 and their product is as large as possible.

2. One number is 4 larger than another. How must they be chosen in order to minimize their product?

3. The sum of two positive integers is 10. Find the maximum value of the sum of their squares.

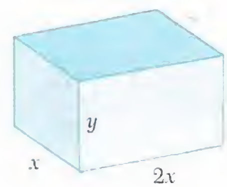
4. Find the minimum possible value of the sum of two positive numbers such that their product is m.

5. Find the value of m if the sum of squares of the roots of $x^2 + (2 - m)x - m - 3 = 0$ is to be minimum.

6. A farmer has 100 m of fencing and wants to build a rectangular pen for his horse. Find the dimensions of the largest area he can enclose.

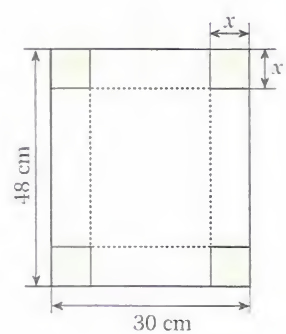
7. A man has 120 metres of fencing. He wishes to enclose a rectangular garden adjacent to a long existing wall. He needs no fence along the wall. What are the dimensions of the largest area he can enclose?

8. A closed rectangular box is to be made with 192 cm^2 of material. The length of its base is twice its width. What is the largest possible volume of such a box?



9. An open tank with a square base is to be made of sheet iron. Its capacity is to be 4 m^3. Find the dimensions of the tank so that the least amount of sheet iron may be used.

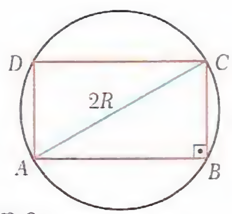
10. A rectangular sheet of tin, 30 cm by 48 cm, has four equal squares cut out at the corners. The sides are then turned up to form an open rectangular box. Find the largest possible volume of the box.



11. A closed cylindrical drum with the volume 54 m^3 is to be manufactured using the minimum amount of metal sheet possible. Find the diameter of the base of the drum.

12. The sum of two non-negative numbers is 10. Find the minimum possible value of the sum of their cubes.

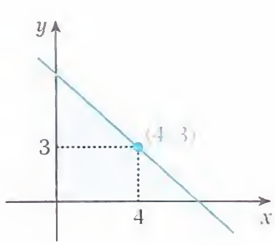
13. Find the area of the largest rectangle that is inscribed in a circle with the radius R. ($[AC]$ is the diameter of the circle because an angle inscribed in a semicircle is a right angle).



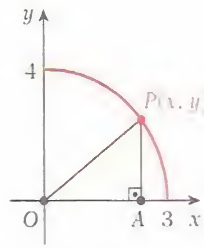
14. What is the shortest distance from a point on the curve $y^2 = 8x$ to the point (4, 2)?

15. Find the point on the curve $y = \ln x$ that is closest to the line $y = x$.

16. What is the minimum possible area of a right triangle that is formed in the first quadrant by a line passing through the point (4, 3) and the coordinate axes?

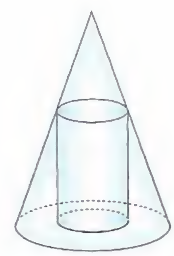


17. In the figure on the right, the point (x, y) lies on the graph of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ in the first quadrant. For what value of y will the area of the triangle POA be maximum?



18. Find the length of the diagonal of the rectangle with the largest area that can be inscribed in an isosceles triangle of base 24 cm and height 10 cm.

19. A right cylinder is inscribed in a right circular cone with the radius 3 cm and height 5 cm. Find the dimensions of the cylinder of maximum volume.



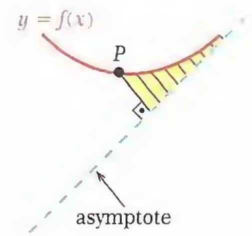
20. Car A is 125 km directly west of car B and begins moving east at 100 km/h. At the same time, car B begins moving north at 50 km/h. At what time t does the minimum distance occur?

21. Among all tangents to graph of $y = x^3 - 6x^2 + \frac{45}{4}x + 1$ at positive x-values, the one which intersects y-axis at maximum y-value is chosen. Find that y-value.

5 PLOTTING GRAPHS

A. ASYMPTOTES

When plotting the graph of a function, we need to know the behavior of the function at infinity and the behavior near points where the function is not defined. To describe these situations, we define the term "asymptote".

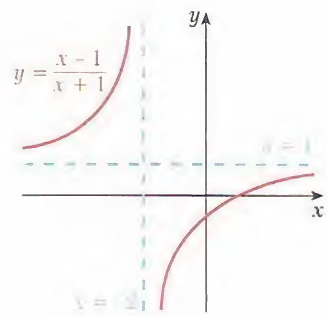


An **asymptote** is a line that a curve approaches closer and closer until the distance between the asymptote and the points on the curve approaches zero.

Consider the graph of the function $f(x)=\dfrac{x-1}{x+2}$.

We represent an asymptote by drawing a dashed line.

Observe that f is increasing on the interval $(-2, \infty)$. So, you might think that its values $f(x)$ increase without bound as x increases without bound. ($f(x) \to \infty$ as $x \to \infty$). However, we can see that the graph of f approaches the line $y = 1$ as $x \to \infty$. So, we say that $y = 1$ is a **horizontal asymptote** of f.



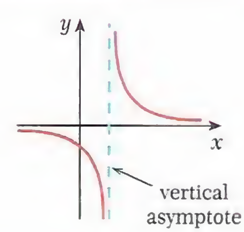
Next, the function f is not defined at $x = -2$. Let us examine the behavior of f near -2. We see that the graph of f goes to plus infinity as x approaches -2 from the left, whereas the graph goes to minus infinity as x approaches -2 from the right. So, we say that $x = -2$ is a **vertical asymptote**.

Definition

vertical asymptote

The line $x = a$ is a **vertical asymptote** of the graph of $f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty.$$



FINDING THE VERTICAL ASYMPTOTE

A rational function $f(x)=\dfrac{P(x)}{Q(x)}$ has a vertical asymptote $x = a$ whenever only the denominator of $f(x)$ equals zero (that is, $Q(a) = 0$ but $P(a) \neq 0$).

Example 49 Find the vertical asymptotes of the function $f(x) = \dfrac{x}{x^2 - 4}$.

Solution Let $P(x) = x$ and $Q(x) = x^2 - 4$.

Note that $x = -2$ and $x = 2$ are the roots of the denominator.

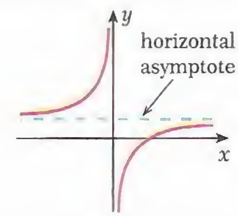
Since $P(-2) \neq 0$ and $P(2) \neq 0$, $x = -2$ and $x = 2$ are both vertical asymptotes of the graph of f.

Definition

horizontal asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of $f(x)$ if either

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$



FINDING THE HORIZONTAL ASYMPTOTE

A rational function $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$ has the following limit as horizontal asympote:

$$\lim_{x \to \mp\infty} f(x) = \begin{cases} \pm\infty & \text{if } n > m \\ a_n / b_m & \text{if } n = m. \\ 0 & \text{if } n < m \end{cases}$$

Example 50 Find the horizontal asymptote of the function $f(x) = \dfrac{x}{x^2 - 4}$.

Solution To find the horizontal asymptotes we must evaluate $\lim\limits_{x \to \infty} f(x)$.

Since the degree of the polynomial in numerator is smaller than the degree of the polynomial in denominator, $\lim\limits_{x \to \infty} \dfrac{x}{x^2 - 4} = 0$. So, $y = 0$ is a horizontal asymptote of f.

Example 51 Find all the vertical and horizontal asymptotes of the function $f(x) = \dfrac{x^2 - x}{2x^2 - 5x + 3}$.

Solution To find the horizontal asymptotes we must evaluate $\lim\limits_{x \to \infty} f(x)$.

Since the degree of $x^2 - x$ equals the degree of $2x^2 - 5x + 3$, $\lim\limits_{x \to \infty} \dfrac{x^2 - x}{2x^2 - 5x + 3} = \dfrac{1}{2}$. So, $y = 1/2$ is a horizontal asymptote of f.

Next, to find the vertical asymptotes we must solve $2x^2 - 5x + 3 = 0$.

$2x^2 - 5x + 3 = (x - 1)(2x - 3) = 0$ gives $x = 1$ and $x = 3/2$.

However, $x = 1$ is also a root of numerator. So, only $x = 3/2$ is a vertical asymptote of f.

Example 52 Find all the vertical and horizontal asymptotes of the function $f(x)=\dfrac{2x^2-3x+5}{x^2+1}$.

Solution $\lim\limits_{x\to\infty}\dfrac{2x^2-3x+5}{x^2+1}=2$. So, $y=2$ is a horizontal asymptote of f.

Since the denominator x^2+1 is never equal to zero, f has no vertical asymptotes.

Example 53 Find all the asymptotes of the function $f(x)=2x^3-5x^2+7x-12$.

Solution The function f is a polynomial function. But we can write it as a rational function with the denominator 1 such as $f(x)=\dfrac{2x^3-5x^2+7x-12}{1}$.

Since the denominator is never equal to zero, f has no vertical asymptotes.

Next, compute $\lim\limits_{x\to\pm\infty}(2x^3-5x^2+7x-12)$.

We know that the limit of a polynomial at infinity is the limit of the term of highest degree.

So, $\lim\limits_{x\to+\infty}(2x^3-5x^2+7x-12)=\lim\limits_{x\to+\infty}2x^3=2\cdot(\infty)^3=\infty$.

$\lim\limits_{x\to-\infty}(2x^3-5x^2+7x-12)=\lim\limits_{x\to-\infty}2x^3=2\cdot(-\infty)^3=-\infty$.

In other words, $\lim\limits_{x\to+\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$ do not exist. Therefore, f has no horizontal asymptote.

Note

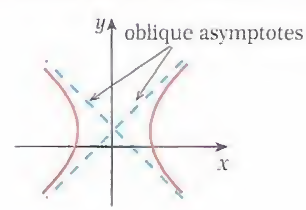
A polynomial function has no vertical or horizontal asymptotes.

Definition

oblique asymptote

The line $y=mx+n$ is an **oblique asymptote** of the graph of $f(x)$ if either

$$\lim_{x\to\infty}[f(x)-(mx+n)]=0 \text{ or } \lim_{x\to-\infty}[f(x)-(mx+n)]=0.$$



To find the equation of an oblique asymptote, we use long division.

For a rational function $f(x)=\dfrac{P(x)}{Q(x)}$ for which the degree of P is exactly one more than the degree of Q, by dividing $Q(x)$ into $P(x)$, we get $f(x)=mx+n+\dfrac{c}{Q(x)}$.

In this case, we have $\lim\limits_{x\to\pm\infty}[f(x)-(mx+n)]=\lim\limits_{x\to\pm\infty}\dfrac{c}{Q(x)}=0$.

So, the line $y=mx+n$ is an oblique asymptote of the graph of $f(x)$.

Example 54 Find all the asymptotes of the graph of $f(x)=\dfrac{x^2+x}{x-2}$.

Solution $x = 2$ is a vertical asymptote of the graph of f because 2 makes only the denominator zero. Note that the degree of the numerator is one more than the degree of the denominator. So, the graph of f has an oblique asymptote.

By long division of $x - 2$ into $x^2 + x$, we can find that $f(x)=\dfrac{x^2+x}{x-2}=x+3+\dfrac{6}{x-2}$.

So, $y = x + 3$ is an oblique asymptote of f.

Check Yourself 12

Find all the asymptotes of the graph of each of the following functions.

1. $f(x)=\dfrac{x-1}{2x+3}$
2. $f(x)=-\dfrac{5x}{3+x^2}$
3. $f(x)=\dfrac{x^2-9}{-3+7x-2x^2}$
4. $f(x)=\dfrac{x^2-x-2}{x-1}$

Answers

1. $x=-\dfrac{3}{2},\ y=\dfrac{1}{2}$
2. $y = 0$
3. $x=\dfrac{1}{2},\ y=-\dfrac{1}{2}$
4. $x = 1,\ y = x$

B. CURVE PLOTTING

Curve plotting is the final part of our study of the derivatives. So far we have seen how to use the derivatives to find the most interesting features of a graph. With the use of all the information about the graph of a function, we can easily draw it.

STEPS OF CURVE PLOTTING

1. **Domain:** Find where $f(x)$ is defined.
2. **Intervals of Increase and Decrease:** Construct the sign chart of $f'(x)$ to determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing.
3. **Local Extrema:** Find the critical points of f and classify each as a maximum, a minimum, or neither by using the First Derivative Test.
4. **Concavity and Inflection Points:** Construct the sign chart of $f''(x)$ to determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down. With the help of the chart, find the inflection points.
5. **Intercepts:** In $y = f(x)$ setting $x = 0$ gives the y-intercept and $y = 0$ gives the x-intercept(s). To find the x-intercept(s) may be difficult, in which case we do not use this information.
6. **Behavior at Infinity:** Consider $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ to see how the graph of f behaves as $x \to \pm\infty$.
7. **Asymptotes:** Find all the asymptotes of the graph and draw the asymptotes in a coordinate plane by using dashed lines.
8. **Graph:** Start graphing by plotting the local extrema, inflection points, and intercepts. Then, using the rest of the information, complete the plot by joining the plotted points.

Example 55 Plot the graph of the function $f(x) = x^3 - 3x - 2$.

Solution 1. Domain: Recall that the domain of a polynomial function is all real numbers. So, f is defined for all the values of x.

2. Intervals of Increase and Decrease: $f'(x) = 3x^2 - 3 = 3(x + 1)(x + 1)$
When $f'(x) = 0$ we have $x = -1$ and $x = 1$.
The sign chart of f' shows that f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		+		−		+	
$f(x)$		↗	(max)	↘	(min)	↗	

3. Local Extrema: We have learned that a polynomial function is differentiable everywhere. So, the critical points of $f(x)$ are the roots of $f'(x) = 0$.

From the results of Step 2, we say that f has a local maximum at $x = -1$ and a local minimum at $x = 1$.

4. Concavity and Inflection Points: $f''(x) = 6x = 0$

When $f''(x) = 0$ we have $x = 0$.

The sign chart of f'' shows that f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

So, f has an inflection point at $x = 0$.

x	$-\infty$		0		$+\infty$
$f''(x)$		−		+	
$f(x)$		concave down		concave up	

5. Intercepts: $x = 0 \Rightarrow y = -2$ (y-intercept)

Setting $y = 0$ leads to a cubic equation. Since the solution is not readily found, we will not use this information.

6. Behavior at Infinity: Recall that the limit of a polynomial function at infinity is the limit of the term of highest degree.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^3 - 3x - 2) = \lim_{x \to -\infty} x^3 = (-\infty)^3 = -\infty$$

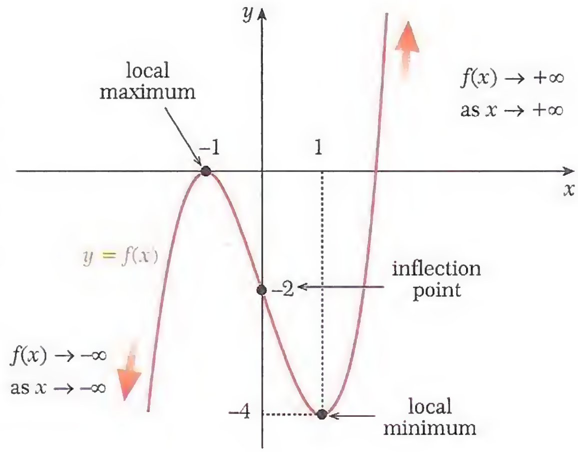
(This means that $f(x)$ decreases without bound as x decreases without bound. So, the graph of f goes to plus infinity as $x \to -\infty$)

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x^3 - 3x - 2) = \lim_{x \to +\infty} x^3 = (+\infty)^3 = +\infty$$

(This means that $f(x)$ increases without bound as x decreases without bound. So, the graph of f goes to plus infinity as $x \to +\infty$)

7. Asymptotes: A polynomial function has no asymptotes.

8. Graph: We can find $f(-1) = 0$, $f(1) = -4$, and $f(0) = -2$. Plot a local maximum at $(-1, 0)$ a local minimum at $(1, -4)$, an inflection point at $(-1, 0)$, and the y-intercept at $y = -2$. Finally, complete the graph by passing a smooth curve through the plotted points.



It is clear from the graph of f that $x = -1$ is a root of $f(x) = 0$. So, $x + 1$ is a factor of $x^3 - 3x - 2$. The other factor can be found by division:

$f(x) = (x + 1)(x^2 - x - 2) = (x + 1)^2(x - 2)$.

Hence $x = 2$ is also a root of $f(x) = 0$ and the graph crosses the x-axis at this point. But note that $x = -1$ is a "double root" of $f(x) = 0$ and the graph is tangent to the x-axis at $x = -1$.

Example 56 Plot the graph of the function $f(x) = -x^4 + 8x^2 - 7$.

Solution

1. Domain: Since f is a polynomial, it is defined for all the values of x.

2. Intervals of Increase and Decrease:

   $f'(x) = -4x^3 + 16x = -4x(x^2 - 4)$

   When $f'(x) = 0$ we have $x = -2$, $x = 0$, and $x = 2$.

   f is increasing on $(-\infty, -2)$ and $(0, 2)$, and decreasing on $(-2, 0)$ and $(2, +\infty)$.

x	$-\infty$	-2		0		2	$+\infty$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	↗	(max)	↘	(min)	↗	(max)	↘

3. Local Extrema: From the sign chart of $f'(x)$, f has local maximum at $x = -2$ and $x = 2$, a local minimum at $x = 0$.

4. Concavity and Inflection Points:

   $f''(x) = -12x^2 + 16 = 0$

   When $f''(x) = 0$ we have $x = \pm\dfrac{2}{\sqrt{3}}$.

   f has inflection points at $x = -\dfrac{2}{\sqrt{3}}$ and $x = \dfrac{2}{\sqrt{3}}$.

x	$-\infty$	$-\frac{2}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$	$+\infty$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	concave down		concave up		concave down

5. Intercepts: $x = 0 \Rightarrow y = -7$ (y-intercept)

   $y = 0 \Rightarrow x_1 = -1, x_2 = 1, x_3 = -\sqrt{7}, x_4 = \sqrt{7}$ (x-intercepts)

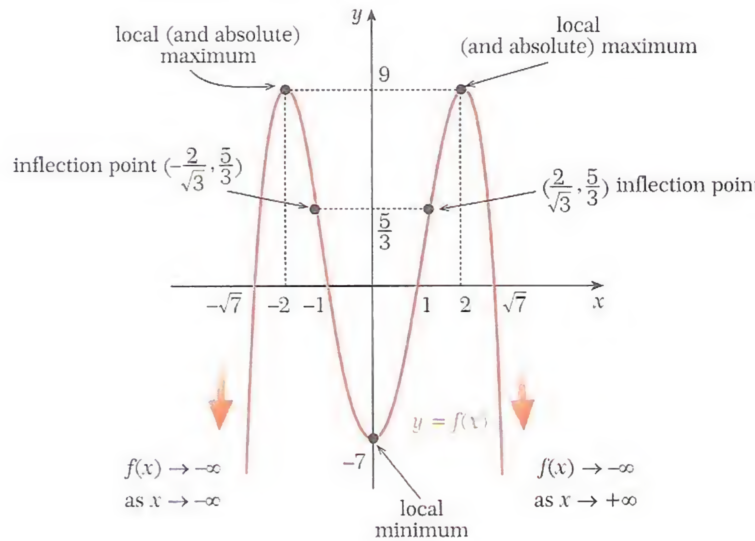
6. Behavior at Infinity:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (-x^4) = -(-\infty)^4 = -\infty \text{ and } \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (-x^4) = -(+\infty)^4 = +\infty$$

The graph goes to $-\infty$ to the left and to the right.

7. Asymptotes: Since f is polynomial, it has no asymptotes.

8. Graph: $f(-2) = 9, f(0) = -7, f(2) = 9, \; f(-\frac{2}{\sqrt{3}}) = \frac{5}{3}, \; f(\frac{2}{\sqrt{3}}) = \frac{5}{3}$.



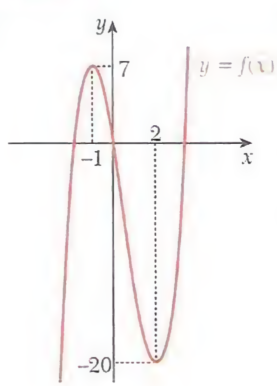
Check Yourself 13

Plot the graph of each function.

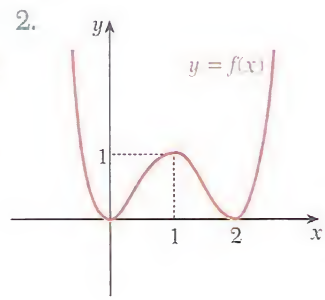
1. $f(x) = 2x^3 - 3x^2 - 12x$ 2. $f(x) = x^2(x - 2)^2$

Answers

1.



2.



Example 57 Plot the graph of the function $f(x)=\dfrac{x-3}{2x+4}$.

Solution

1. Domain: Recall that the domain of a rational function is all real numbers except the numbers that make the denominator zero. So, f is defined everywhere except $x = -2$.

2. Intervals of Increase and Decrease: $f'(x)=\dfrac{1\cdot(2x+4)-(x-3)\cdot 2}{(2x+4)^2}=\dfrac{10}{(2x+4)^2}$

   Since $f'(x) > 0$ for all the values of x except -2. So, f is always increasing in its domain.

3. Local Extrema: Note that f' does not change its sign. By the first derivative test, we say that f has no local extrema.

4. Concavity and Inflection Points: $f''(x)=\dfrac{-40}{(2x+4)^3}$

   The sign chart of f'' shows that f is concave up on $(-\infty, -2)$ and concave down on $(-2, -\infty)$. Observe that f'' changes its sign at $x = -2$. But at this point f is not defined. Therefore, there is no inflection point.

x	$-\infty$	-2	$+\infty$
$f''(x)$	$+$	o	$-$
$f(x)$	concave up		concave down

5. Intercepts: $x=0 \Rightarrow y=-\dfrac{3}{4}$ (y-intercept)

   $y=0 \Rightarrow x=3$ (x-intercept)

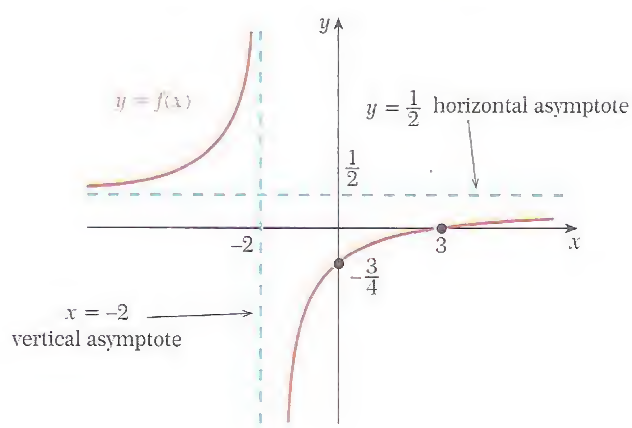
6. Behavior at Infinity:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{x-3}{2x+4}=\frac{1}{2}.$$

7. Asymptotes: From Step 6, $y=\dfrac{1}{2}$ is a horizontal asymptote of the graph of f.

Also, $x = -2$ is a vertical asymptote of the graph of f because -2 makes the denominator zero.

8. Graph:



Example 58 Plot the graph of the function $f(x)=\dfrac{x}{x^2-4}$.

Solution

1. Domain: The domain of f is all real numbers except $x = -2$ and $x = 2$.

2. Intervals of Increase and Decrease: $f'(x)=\dfrac{1\cdot(x^2-4)-x\cdot 2x}{(x^2-4)^2}=\dfrac{-x^2-4}{(x^2-4)^2}=\dfrac{-(x^2+4)}{(x^2-4)^2}$

   Since $f'(x) < 0$ for all the values of x except -2 and 2, f is always decreasing in its domain.

3. Local Extrema: f has no local extrema.

4. Concavity and Inflection Points:

   $$f''(x)=\frac{2\cdot x(x^2+8)}{(x^2-4)^3}$$

   $f''(x) = 0$ only when $x = 0$.

   f'' is not defined at $x = -2$ and $x = 2$.

x	$-\infty$ … -2	… 0	… 2	… $+\infty$
$f''(x)$	−	+	−	+
$f(x)$	concave down	concave up	concave down	concave up
		(inf)		

   Thus, f is concave up on $(-2, 0)$ and $(2, \infty)$ and concave down on $(-\infty, -2)$ and $(0, 2)$. The sign of $f''(x)$ changes at the points $x = -2$, 0, and 2. But the only inflection point is $x = 0$ because f is not defined at -2 and 2.

5. Intercepts: $x = 0 \Rightarrow y = 0$ and $y = 0 \Rightarrow x = 0$.

   The point $(0, 0)$ is the only intercept.

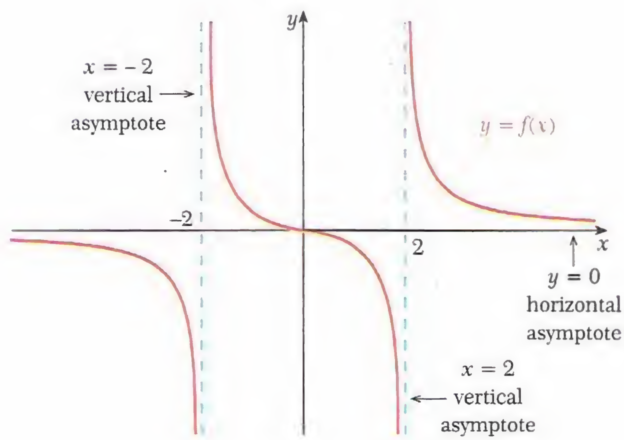
6. Behavior at Infinity:

   $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{x}{x^2-4}=0.$$

7. Asymptotes: From Step 6, $y = 0$ (the x – axis) is a horizontal asymptote of the graph of f.

   Next, $x = -2$ and $x = 2$ are vertical asymptotes of the graph of f because they make the denominator zero.

8. Graph:



Example 59 Plot the graph of the function $f(x) = \dfrac{x^2 - x + 4}{x - 1}$.

Solution 1. Domain: f is defined for all the values of x except $x = 1$.

2. Intervals of Increase and Decrease: $f'(x) = \dfrac{(2x-1)\cdot(x-1)-(x^2-x+4)}{(x-1)^2} = \dfrac{x^2-2x-3}{(x-1)^2}$

When $f'(x) = 0$, $x = -1$ and $x = 3$.

Also, note that $f'(x)$ is not defined at $x = 1$.

From the sign chart of f', f is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing on $(-1, 1)$ and $(1, 3)$.

x	$-\infty$		-1		1		3		$+\infty$
$f'(x)$		+	0	−	‖	−	0	+	
$f(x)$		↗	(max)	↘	‖	↘	(min)	↗	

3. Local Extrema: f has a local maximum at $x = -1$ and a local minimum at $x = 3$.

4. Concavity and Inflection Points: $f''(x) = \dfrac{8}{(x-1)^3}$

We conclude that f is concave down on $(-\infty, -1)$ and concave up on $(1, \infty)$. But it has no inflection point because -1 is not in the domain of f.

x	$-\infty$	1	$+\infty$
$f''(x)$	−	‖	+
$f(x)$	concave down	‖	concave up

5. Intercepts: $x = 0 \Rightarrow y = -4$ (y-intercept)

$y = 0 \Rightarrow x^2 - x + 4 = 0 \Rightarrow \Delta < 0$ (no x – intercepts)

6. Behavior at Infinity: $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{x^2 - x + 4}{x - 1} = \pm\infty$.

7. Asymptotes: Note that the degree of the numerator of f is exactly one more than the degree of the denominator of f.

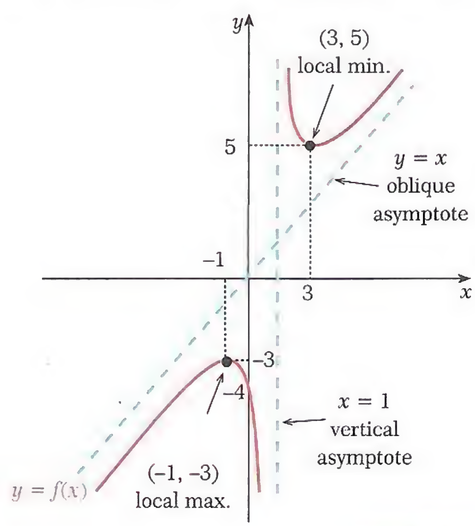
So, f has an oblique asymptote.

By long division, we have $f(x) = x + \dfrac{4}{x-1}$.

So, $y = x$ is an oblique asymptote of f.

Next, $x = 1$ is a vertical asymptote of f.

8. Graph: $f(-1) = -3$ and $f(3) = 5$.



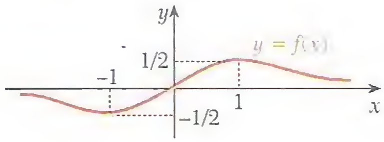
Check Yourself 14

Plot the graph of each function.

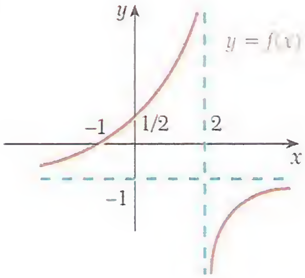
1. $f(x) = \dfrac{x}{x^2+1}$ 2. $f(x) = \dfrac{x+1}{2-x}$

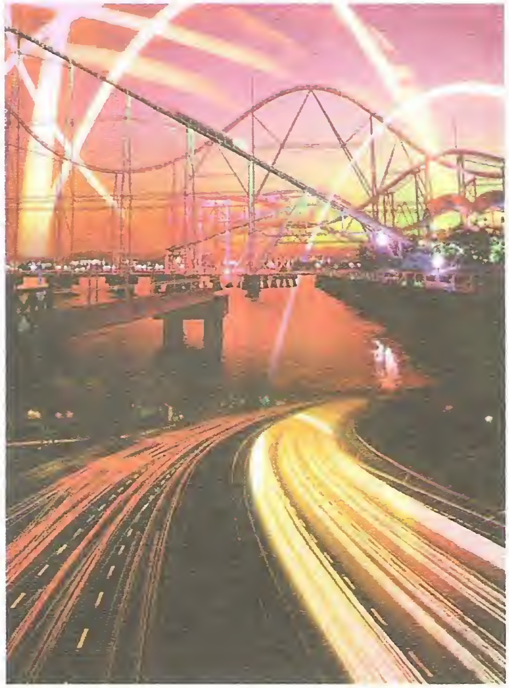
Answers

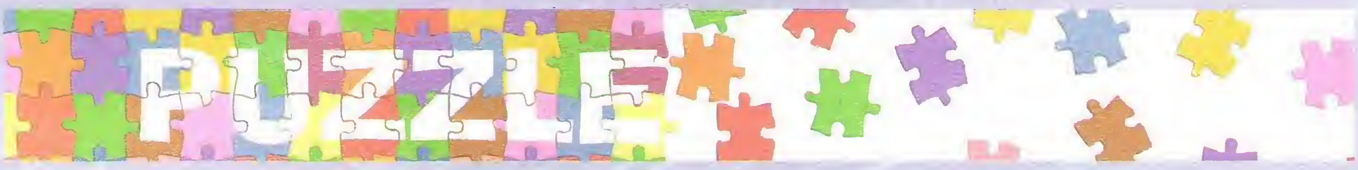
1.

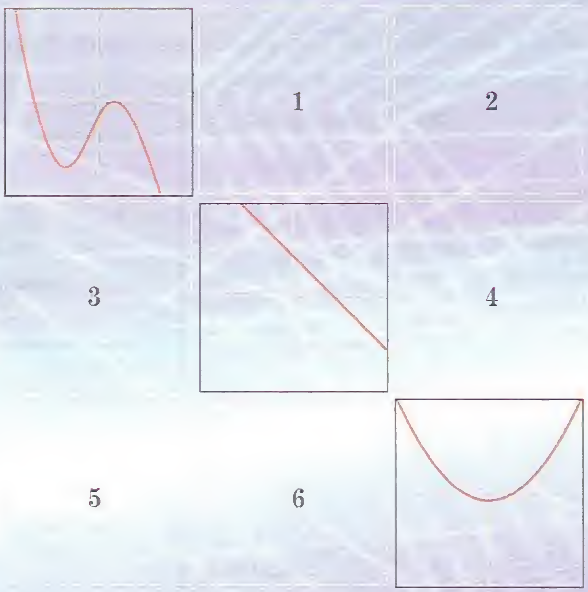


2.

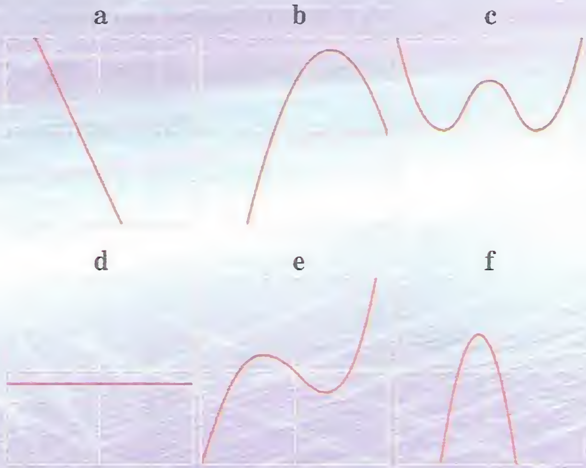








Fill in the 3 × 3 field of squares such that the graph of the derivative is located below each graph.



EXERCISES 2.5

A. Asymptotes

1. Find all the asymptotes of the graph of each function.

a. $f(x)=\dfrac{2}{x+1}$

b. $f(x)=\dfrac{1}{(x-1)^3}$

c. $f(x)=x^3-4x^2-5x+6$

d. $f(x)=\dfrac{3x+2}{4-x}$

e. $f(x)=\dfrac{-3x}{(x+3)^2}$

f. $f(x)=\dfrac{x^2+1}{1-x^2}$

g. $f(x)=\dfrac{(x-1)^2}{x^2+2x-3}$

h. $f(x)=\dfrac{4-x-3x^2}{2x^2+3x-9}$

i. $f(x)=\dfrac{x-3}{x^2-5x-6}$

j. $f(x)=\dfrac{x^3}{x^2+9}$

k. $f(x)=\dfrac{x^2+2x+3}{x-1}$

2. The curve $y=\dfrac{3x-1}{x^2+x+m}$ has exactly one vertical asymptote. Find m.

B. Curve Plotting

3. Plot the graph of each polynomial function.

a. $f(x)=x^3-6x^2$

b. $f(x)=(x-1)^2(x+3)$

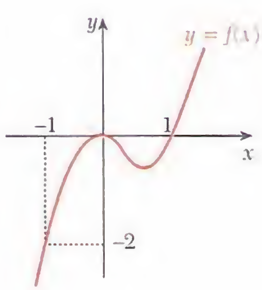
c. $f(x)=x^3-2x^2+x-2$

d. $f(x)=(x^2-4)(3-x)$

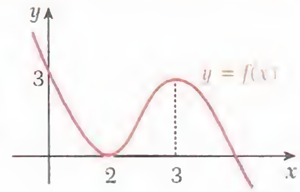
e. $f(x)=x^4-2x^2+1$

f. $f(x)=x(x-1)(x+1)^2$

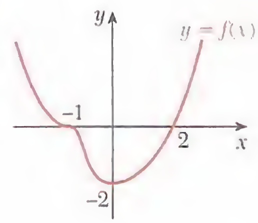
4. In the figure the graph of a cubic function $y=f(x)$ is given. Find the local minimum value of f.



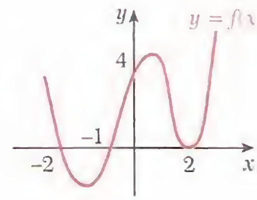
5. The graph of the function $f(x)=a(x-2)^2(x+b)$ is shown in the figure. Find a and b.



6. The graph of the function $f(x)=a(x+b)^3(x+c)$ is shown in the figure. f has an inflection point at $x=-1$. Find the sum $a+b+c$.



7. Find the equation of a polynomial function of degree 4 whose graph is shown in the figure.



8. Plot the graph of each rational function.

a. $f(x)=\dfrac{x-1}{x+1}$

b. $f(x)=\dfrac{1}{x^2+1}$

c. $f(x)=\dfrac{x-1}{x^2-2x-3}$

d. $f(x)=\dfrac{(x+1)^2}{x^2-4x+3}$

e. $f(x)=\dfrac{x^2-9}{x^2+3x}$

f. $f(x)=\dfrac{x^2+x}{x-2}$

Mixed Problems

9. Plot the graph of each function.

a. $f(x)=-x\sqrt{4-x^2}$

b. $f(x)=\sqrt{\dfrac{x-2}{x+2}}$

c. $f(x)=e^{\frac{x-1}{x-2}}$

d. $f(x)=\dfrac{-x}{e^x}$

e. $f(x)=\ln(x^2+4)$

f. $f(x)=\sin x+\cos^2 x$

10. State how many solutions the equation $2x^3+3x^2-12x+3=a$ has for each value of a.

CHAPTER SUMMARY

1. L'Hospital's Rule

- If we have the indeterminate forms 0/0 or ∞/∞.

  L'Hospital's Rule says that if $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then

  $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.

2. Applications of the First Derivatives

- A function f is **increasing** on an interval I if $f(x)$ increases as x increases for every x in I.

  Similarly, f is decreasing on an interval I if $f(x)$ **decreases** as x increases for every x in I.

- If $f'(x) > 0$ for all the values of x in the interval I, then $f(x)$ is increasing in I.

  If $f'(x) < 0$ for all the values of x in the interval I, then $f(x)$ is decreasing in I.

- A function f has an **absolute maximum** at c if

  $f(c) \geq f(x)$ for all the values of x in the domain of f.

  Similarly, f has an **absolute minimum** at c if

  $f(c) \leq f(x)$ for all the values of x in the domain of f.

- A function f has an **local maximum** at c if

  $f(c) \geq f(x)$ for all the values of x in an interval I containing c.

  Similarly, f has a **local minimum** at c if

  $f(c) \leq f(x)$ for all the values of x in an interval I containing c.

- The number c in the domain of f is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ does not exist.

- If a function f has a local extremum at c, then c is a critical point of f.

- Let c be a critical point of a function f. The **First Derivative Test** says that:

  1. if f' changes from positive to negative at c, then f has a local maximum at c.
  2. if f' changes from negative to positive at c, then f has a local minimum at c.
  3. If f' does not change sign at c, then f has no local maximum or minimum at c.

- If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

- To find the absolute extrema of a continuous function on a closed interval $[a, b]$, use the **Closed Interval Method**.

  1. Find the critical points of f on the interval $[a, b]$.
  2. Evaluate $f(x)$ at each critical point.
  3. Evaluate $f(a)$ and $f(b)$.
  4. The largest of the values of f found in previous is the absolute maximum, shown by $f_{max}[a, b]$; the smallest of these values is the absolute minimum, $f_{min}[a, b]$.

3. Applications of the Second Derivatives

- A function f is **concave up** on an interval I if the graph of f lies above all of its tangent lines on the interval I.

  Similarly, f is **concave down** on I if the graph of f lies below all of its tangent lines on I.

- If $f''(x) > 0$ for all the values of x on the interval I, then f is concave up on I.

  If $f''(x) < 0$ for all the values of x on the interval I, then f is concave down on I.

- An **inflection point** is a point where a graph changes its direction of concavity.

- To find the inflection points of a function, follow the steps:

  1. Find the points where $f''(x) = 0$ and $f''(x)$ does not exist. These points are the possible inflection points of the function f.
  2. Construct the sign chart of $f''(x)$. If the sign of $f''(x)$ changes across the point $x = a$, then $(a, f(a))$ is an inflection point of f.

- The **Second Derivative Test** is an alternative test for determining whether a critical point of f is a local maximum or a local minimum. Let f be twice differentiable on an interval I and c be a critical point of f in I such that $f'(c) = 0$.

  1. If $f''(c) > 0$, then f has a local minimum at $x = c$.
  2. If $f''(c) < 0$, then f has a local maximum at $x = c$.

4. Optimization Problems

- To solve optimization problems, follow the steps:

  1. Determine the quantity to be maximized or minimized and label it with a letter (say M for now).
  2. Assign letters for other quantities, possibly with the help of a figure.
  3. Express M in terms of some of the other variables.

4. Use the information given in the problem to write M as a function of one variable x, say $M = M(x)$.
5. Find the domain of the function $M(x)$.
6. Find the maximum (or minimum) value of $M(x)$.

An **asymptote** is a line that a curve approaches more and more closely until the distance between the asymptote and the points on the curve must approach zero.

The line $x = a$ is a **vertical asymptote** of the graph of $f(x)$ if either $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} f(x) = \pm\infty$.

A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ has a vertical asymptote $x = a$ whenever only the denominator of $f(x)$ equals zero (that is, $Q(a) = 0$ but $P(a) \neq 0$).

The line $y = b$ is a **horizontal asymptote** of the graph of $f(x)$ if either $\lim\limits_{x \to -\infty} f(x) = b$ or $\lim\limits_{x \to +\infty} f(x) = b$.

To find the horizontal asymptote of a rational function, apply the rule:

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \begin{cases} \pm\infty, & n > m \\ a_n / b_m, & n = m \\ 0, & n < m \end{cases}$$

The line $y = mx + n$ is an **oblique asymptote** of the graph of $f(x)$ if either

$$\lim_{x \to \infty}[f(x) - (mx + n)] = 0 \text{ or } \lim_{x \to -\infty}[f(x) - (mx + n)] = 0.$$

If the degree of the numerator of a rational function is exactly one more than the degree of its denominator, then the graph of the function has an oblique asymptote.

A polynomial function has no asymptotes.

To plot the graph of a function, follow the steps:

1. **Domain:** Find where $f(x)$ is defined.
2. **Intervals of Increase and Decrease:** Construct the sign chart of $f'(x)$ to determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing.
3. **Local Extrema:** Find the critical points of f and classify each as a maximum, a minimum, or neither by using the First Derivative Test.
4. **Concavity and Inflection Points:** Construct the sign chart of $f''(x)$ to determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down. With the help of the chart, find the inflection points.
5. **Intercepts:** In $y = f(x)$ setting $x = 0$ gives the y-intercept and $y = 0$ gives the x-intercept(s). To find the x-intercept(s) may be difficult, in which case we do not use this information.
6. **Behavior at Infinity:** Find $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ to see how the graph of f behaves as $x \to \pm\infty$.
7. **Asymptotes:** Find all the asymptotes of the graph and draw the asymptotes in a coordinate plane by using dashed lines.
8. **Graph:** Start graphing by plotting the local extrema, inflection points, and intercepts. Then, using the rest of the information, complete the plot by joining the plotted points.

Concept Check

- What does L'Hospital's Rule say? How do we know how many times to use it in a given problem?
- How can we use L'Hospital's Rule if we have the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$?
- What is the relation between the first derivative and the increasing and decreasing behavior of the function?
- Explain the difference between an absolute extremum and a local extremum.
- Define a critical point of a function.
- How can we determine that a critical point of a function is a maximum, a minimum, or neither?
- In what conditions does a function have both the absolute maximum and absolute minimum?
- Discuss the significance of the sign of the second derivative.
- What is an inflection point?
- What does a curve $y = f(x)$ on which $f' < 0$ and $f'' > 0$ look like?
- Let n be a positive integer. For what values of n does the function $f(x) = x^n$ have an inflection point at the origin?
- What are the relative advantages and disadvantages of the first and second derivative tests?
- Describe the procedure to solve optimization problems.
- What is an asymptote?
- How many different asymptotes are there?
- Give an example of a function whose graph has an oblique asymptote.
- Describe the procedure to plot the graph of a function.
- When plotting the graph of a function, how do we know the behavior of the function at infinity?

CHAPTER REVIEW TEST 2A

1. Find $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-4x+3}$.

A) 1 B) $-\dfrac{3}{2}$ C) 0 D) $-\dfrac{3}{4}$ E) $\dfrac{2}{3}$

2. Find $\lim\limits_{x\to \pi}\dfrac{1-\sin\dfrac{x}{2}}{\pi - x}$.

A) 1 B) $\dfrac{1}{2}$ C) 0 D) $-\dfrac{1}{2}$ E) -1

3. Find the interval on which $f(x) = x^2 - 6x + 2$ is decreasing.

A) $(-\infty, 3)$ B) $(-3, 3)$ C) $(3, \infty)$ D) $(0, \infty)$ E) $(-\infty, 6)$

4. Find the maximum value of the function $f(x) = e^{4x - x^2}$.

A) 1 B) e C) e^2 D) e^4 E) e^8

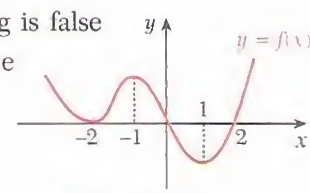
5. For what values of k is $f(x) = x^3 + (k+1)x^2 + 3x + 2$ always increasing?

A) $-6 < k < 3$ B) $k > 0$ C) $-4 < k < 0$ D) $-3 < k < 2$ E) $-4 < k < 2$

6. Which of the following is a local extremum of $f(x) = x^3 - 3x^2 + 3x + 1$?

A) -6 B) -1 C) 0 D) 2 E) no extremum

7. Which of the following is false for the graph of the function $y = f(x)$?



A) $f(2) = 0$ B) $f'(-1) = 0$ C) $f''(1) > 0$ D) $f''(-1) > 0$ E) $f'(0) < 0$

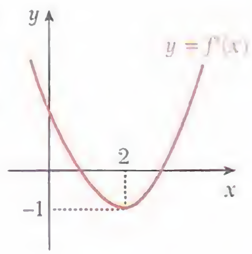
8. Find the interval on which $y = (x+2)^3$ is concave up.

A) $(-2, \infty)$ B) $(-\infty, -2)$ C) $(-2, 2)$ D) $(-\infty, \infty)$ E) $(2, \infty)$

9. For what value of m does the polynomial $P(x) = x^4 + x^3 + (m - 1)x^2$ have an inflection point at $x = -1$?

A) -3 B) -2 C) -1 D) 0 E) 1

10. The graph of the derivative of the function $f(x) = x^3 + ax^2 + bx + 1$ is given in the figure. Find $a + b$.



A) 17 B) 11 C) 5 D) -17 E) -10

11. Find the intersection point of the asymptotes of $y = \dfrac{3-x}{x+2}$.

A) $(-2, 3)$ B) $(3, -1)$ C) $(-2, -1)$ D) $(2, 1)$ E) $(-1, 3)$

12. Which of the following is true for the function $f(x) = 2x^3 + 3x^2 + 12x + 4$?

A) f has a local minimum at $x = 0$.
B) $f'(2) < 0$
C) f is concave up on $(-\infty, -\frac{1}{2})$.
D) f is always increasing.
E) f has a local maximum at $x = -1$.

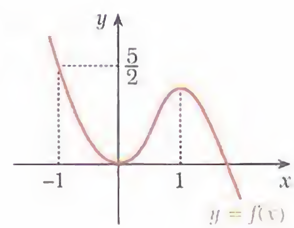
13. Let x_1 and x_2 be the roots of the equation $y = x^2 - (m + 1)x + 2m - 1 = 0$. Find the value of m that minimizes $x_1^2 + x_2^2$.

A) 0 B) 1 C) -1 D) 2 E) -2

14. Find the point on the parabola $y = \dfrac{x^2}{2}$ that is closest to the point $(-\frac{3}{2}, 0)$.

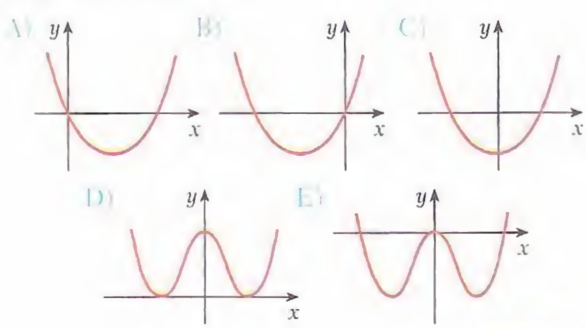
A) $(-1, \frac{1}{2})$ B) $(1, \frac{1}{2})$ C) $(0, 0)$ D) $(-\frac{1}{2}, \frac{1}{8})$ E) $(\frac{1}{2}, \frac{1}{8})$

15. Given the graph of a cubic function f, find $f(2)$.



A) $\frac{3}{2}$ B) 0 C) -1 D) -2 E) $-\frac{5}{2}$

16. Which one of the following graphs could be the graph of $y = x^4 - 2x^2$?



1. Find the value of the limit $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x^2-1}$.

A) 0 B) $\dfrac{1}{2}$ C) 1 D) $\dfrac{1}{3}$ E) $\dfrac{1}{6}$

2. Find the value of the limit $\lim\limits_{x\to\infty}\dfrac{3x^3+5x+4}{e^x+3x}$.

A) 3 B) 0 C) $\dfrac{3}{e}$ D) ∞ E) $\dfrac{5}{3}$

3. Let f be an increasing function on the closed interval [–4, 4]. Which of the following is definitely true?

A) $f(3) > 0$ B) $f(-2) < 0$

C) $f'(1) < 0$ D) $f(2) < f(-2)$

E) $f(-1) < f(1)$

4. If the function $f(x)=\dfrac{x^2+x}{x+a}$ has a local extremum at $x = 2$, find a.

A) $\dfrac{4}{5}$ B) $\dfrac{2}{5}$ C) –1 D) $-\dfrac{4}{5}$ E) $-\dfrac{2}{5}$

5. Find the local minimum value of $f(x)=x+\dfrac{8}{x^4}$.

A) 1 B) $\dfrac{5}{2}$ C) $\dfrac{3}{4}$ D) $\dfrac{5}{4}$ E) $\dfrac{3}{2}$

6. Given that $f(x) = a\ln x + bx^2 + x - 2$ has a local minimum at $x = 1$ and a local maximum at $x = 2$, find $a + b$.

A) $-\dfrac{5}{6}$ B) $-\dfrac{2}{3}$ C) $-\dfrac{1}{6}$ D) $-\dfrac{5}{3}$ E) $-\dfrac{4}{3}$

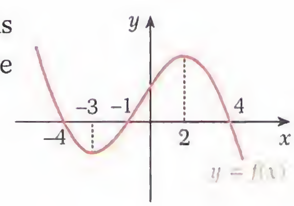
7. Find the sum of the maximum and minimum values of $f(x)=\dfrac{x-2}{x-3}$ on the interval [–4, 1].

A) $\dfrac{19}{14}$ B) $\dfrac{9}{7}$ C) 1 D) $\dfrac{17}{14}$ E) $\dfrac{8}{7}$

8. Given that $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}=2$, find $\dfrac{a}{b}$.

A) 0 B) 1 C) e D) e^2 E) $e^2 + 1$

9. Which of the following is false for the graph of the function $y = f(x)$?



A) $f'(2) = 0$ B) $f''(\frac{11}{5}) < 0$

C) $f(-1) = 0$ D) $f'(1) > 0$

E) $f''(-\frac{13}{4}) < 0$

10. If $x = a$ and $y = b$ are the asymptotes of the graph of $f(x)=\dfrac{x^2+3x+2}{x^2-2x-3}$, find $a + b$.

A) 1 B) 2 C) 3 D) 4 E) 5

11. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 3.

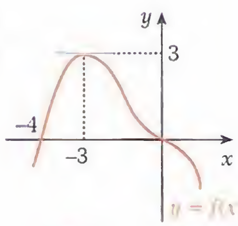
A) $2\sqrt{2}$ B) $3\sqrt{2}$ C) 9

D) $8\sqrt{2}$ E) $9\sqrt{2}$

12. Find the maximum possible area of a right triangle whose hypotenuse is 6 cm.

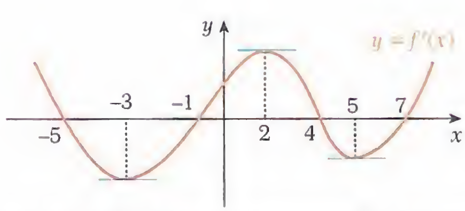
A) 36 B) 12 C) 3 D) 18 E) 9

13. Given the graph of the function f, find the slope of the tangent line to the graph of $g(x) = x \cdot f(x)$ at $x = -3$.



A) 3 B) –4 C) –6 D) 0 E) –3

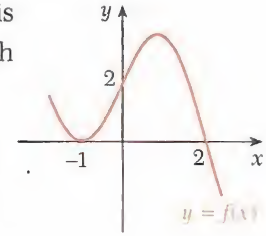
14.



Given the graph of the derivative of a function f, what is the sum of the abscissa of inflection points of f?

A) –9 B) –6 C) 4 D) 7 E) 16

15. Which of the following is the function whose graph is given in the figure?



A) $f(x) = (x + 1)^2(x - 2)$

B) $f(x) = (x + 1)(2 + x)$

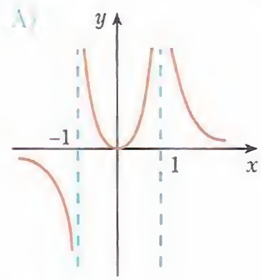
C) $f(x) = (x + 1)(2 - x)$

D) $f(x) = (x + 1)^2(2 - x)$

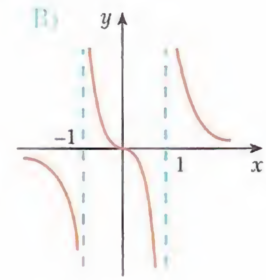
E) $f(x) = (x - 1)^2(x - 2)$

16. Which one of the following could be the graph of the function $f(x) = \dfrac{2x}{x^2 - 1}$?

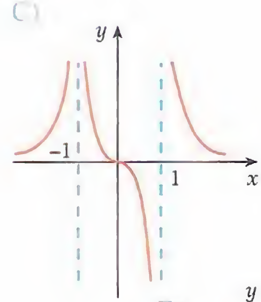
A)



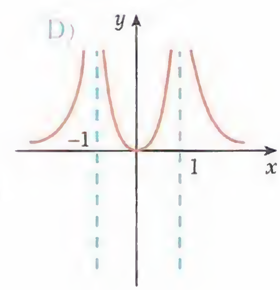
B)



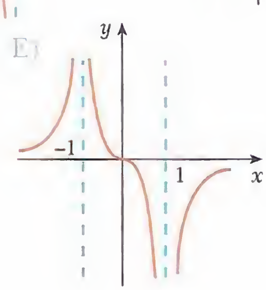
C)



D)



E)



1. Find $\lim_{x\to 1} \dfrac{\ln(2x^2-1)}{\sin(x-1)}$.

A) 2 B) 4 C) 8 D) 16 E) 32

2. Given that the function $f(x) = x^4 - ax^3 + bx^2 - 2x + 3$ has a local minimum at the point (1, 2), find a.

A) –2 B) –1 C) 0 D) 1 E) 2

3. Find the interval on which $y = x\ln x$ is increasing.

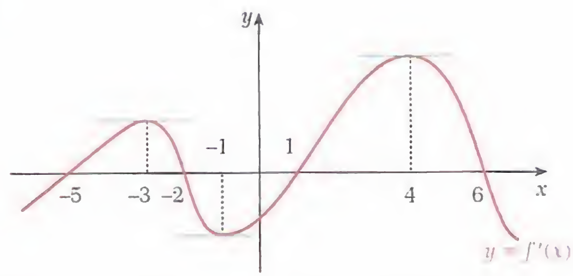
A) $(-\infty, -e)$ B) $(-e, 0)$ C) $(1, e)$ D) $(-\infty, \frac{1}{e})$ E) $(\frac{1}{e}, \infty)$

4. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the extremum points of $y = \dfrac{x^2+1}{x}$. Find the distance between these points.

A) $\sqrt{5}$ B) $2\sqrt{5}$ C) $3\sqrt{5}$ D) $4\sqrt{5}$ E) $5\sqrt{5}$

5. Find m if the inflection point of the function $f(x) = \frac{1}{3}x^3 - x^2 + mx + \frac{2}{3}$ is on the parabola $y = x^2 - 2x + 3$.

A) 2 B) 1 C) 0 D) –1 E) –2

6. 

The graph of the derivative of the function f is given in the figure. For what value of x does f have a local maximum?

A) –5 B) –4 C) –3 D) –2 E) –1

7. At what point does the tangent to the graph $f(x) = x^3 - 3x^2 - 4x - 5$ have the smallest slope?

A) (–2, 7) B) (–1, 1) C) (0, –5) D) (1, –11) E) (2, 7)

8. The length of a line segment [AB] is 10 cm. Take a point C on [AB]. What should the value of |AC| so that $|AC|^2 + |BC|$ is a minimum?

A) $\frac{1}{5}$ B) $\frac{1}{4}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

9. Which one of following could be the value of m if the function $f(x)=\dfrac{x^2-1}{mx+3}$ has no local extrema?

A) -2 B) -1 C) 0 D) 2 E) 4

10. The function $f(x)$ is a positive valued increasing function on the interval (a, b). Which one of the following functions is decreasing on the same interval?

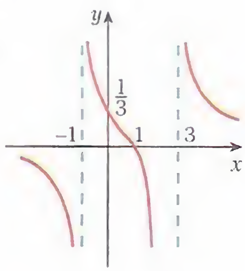
A) $f^2(x)$ B) $-\dfrac{1}{f(x)}$ C) $f^3(x)$

D) $x + f(x)$ E) $\dfrac{1}{f^2(x)}$

11. Given that $y = 3x - 3$ is an oblique asymptote of the graph of $y=\dfrac{(m-3)x^3-4x+1}{x^2+x-3}$, find m.

A) 3 B) 4 C) 5 D) 6 E) 7

12. Which of the following is the function whose graph is given in the figure?



A) $y=\dfrac{x+1}{x^2-2x-3}$ B) $y=\dfrac{x-1}{x^2-2x-3}$

C) $y=\dfrac{x+1}{x^2+2x+3}$ D) $y=\dfrac{x+1}{x^2+3}$

E) $y=\dfrac{-1}{x^2-2x-3}$

13. Let $f(x) = (x - a)(x - b)(x - c)$ and $a < b < c$. Which one of the following statements is false?

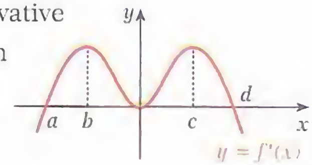
A) $f'(a) > 0$ B) $f'(b) < 0$ C) $f'(c) > 0$

D) $f''(a) < 0$ E) $f''(c) < 0$

14. A right cone is inscribed in a sphere of diameter 6 cm. What is the maximum possible volume of the cone?

A) $\dfrac{32\pi}{3}$ B) $\dfrac{24\pi}{3}$ C) $\dfrac{16\pi}{3}$ D) $\dfrac{12\pi}{3}$ E) $\dfrac{8\pi}{3}$

15. The graph of the derivative of a function f is given in the figure. Which of the following statements is false?



A) $f'(a) = 0$

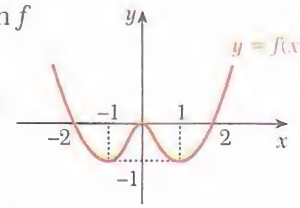
B) $f''(0) = 0$

C) f has a local minimum at $x = a$.

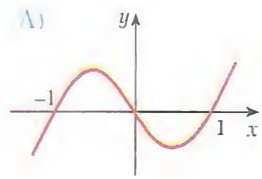
D) f has a local maximum at $x = d$.

E) f has a local maximum at $x = 0$.

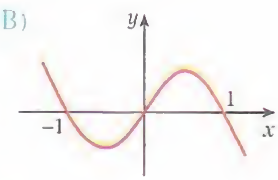
16. The graph of a function f is given in the figure. Which of the following could be the graph of its derivative?



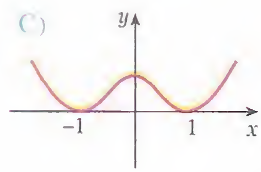
A)



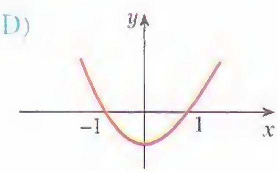
B)



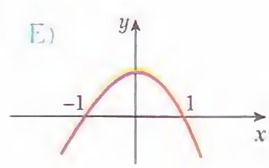
C)



D)



E)



ANSWERS TO EXERCISES

EXERCISES 1.1

1. a. 5 b. –7 c. –2 d. –2 e. 0 f. $\frac{5}{4}$ g. $-\frac{1}{4}$ h. $\frac{4}{9}$ 2. a. $y = 2x + 5$ b. $y = 3x$ c. $y = 11x - 16$ d. $y=\frac{1}{2}x+2$ 3. –2 4. 30; 40 5. $\frac{16\pi}{9}$ 6. 9;11 7. a. rate of change of fuel consumption with respect to change in velocity b. if velocity increases by 20 km/h, oil consumption decreases by 0.05 litres per hour 8. a. x^{10}; 1 b. $\sqrt[3]{x}$; 8 c. 3^x; 4 d. $\cos x$; π

9. a. $2x - 2$ b. $-\frac{3}{(x-1)^2}$ c. $-\frac{3}{2x\sqrt{x}}$ d. $\frac{3}{2\sqrt{3x+1}}$ 10. no; $f'(8^-) \neq f'(8^+)$ 11. does not exist

12. –2, 0, 6 : discontinuity; 1, 3 : corner 13. no; discontinuity at $x = 1$ 14. does not exist 15. D, E, C, A, B

16. –2 17. (–2, 8), (2, 8) 18. a. 50 b. 60 19. $a = 1, b = 6, c = 0$ 20. 0 21. 9.95

EXERCISES 1.2

1. a. 0 b. 0 c. 0 d. $\frac{2}{3}x^7$ e. $1.6x^{-0.2}$ f. $x^{-\frac{1}{5}}$ g. 0 h. $0.21x^{-0.3}$ i. $-84x^{-13}$ j. $10x - 3$ k. $2x+2+\frac{1}{x^2}$ l. $-16t^{-5} + 9t^{-4} - 2t^{-2}$

m. $\frac{1}{2\sqrt{x}}+\frac{1}{3\sqrt[3]{x^2}}+\frac{1}{5\sqrt[5]{x^4}}$ n. $\frac{1}{2\sqrt{x}}-\frac{1}{x\sqrt{x}}-\frac{1}{x^2}$ o. $\frac{1}{x^2}-\frac{3}{2x\sqrt{x}}$ 2. a. $15x^2 - 5$ b. $12x + 1$ c. $-300x - 20$ d. $4x^3 + 3x^2 - 1$

e. $5x^4 - 4x^3 + 9x^2 - 6x + 2$ f. $5t^{\frac{3}{2}} + 4t - \frac{3}{2}t^{-\frac{1}{2}}$ g. $-\frac{6}{(2x+4)^2}$ h. $\frac{3}{(2x+1)^2}$ i. $-\frac{5}{(1+3x)^2}$ j. $\frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}$

k. $\frac{x^2-2x-2}{(x^2+x+1)^2}$ l. $-\frac{3\sqrt{3x}+2\sqrt{x}+\sqrt{3}}{2\sqrt{x}(3x-1)^2}$ 3. a. 8 b. 2 c. –9 d. 1 4. $\frac{1}{2\sqrt{x}}-3$ 5. a. –1 b. 8 c. –8 d. 8

6. a. $6(3x - 1)$ b. $10x(x^2 + 2)^4$ c. $7(x^5- 3x^2+6)^6(5x^4- 6x)$ d. $-3(x-2)^{-4}$ e. $-\frac{40x+12}{(5x^2+3x-1)^3}$ f. $-\frac{4x}{\sqrt{(4x^2+1)^3}}$

g. $3(\sqrt{x+1}+\sqrt{x})^2\cdot(\frac{1}{2\sqrt{x+1}}+\frac{1}{2\sqrt{x}})$ h. $\frac{4(x-1)^4(4x+1)}{(3x+1)^{\frac{2}{3}}}$ i. $-\frac{(3x-1)^6(18x+29)}{(2x+1)^5}$ j. $\frac{405(x-3)^2}{8(x+2)^4}$

k. $\frac{5}{2(2x-1)(3x+1)}\sqrt{\frac{2x-1}{3x+1}}$ l. $3+\frac{3}{4}[2x^2+(x^3+1)^2]^{-\frac{1}{4}}[4x+2(x^3+1)3x^2]$ 7. –12 8. a. $8x + 4$ b. $2x$ c. $\frac{-3x^2}{(x^3-1)^2}$

d. $\frac{(2x-1)(x^2-x-1)}{2(x^2-x)\sqrt{x^2-x}}$ 9. a. 6 b. $14(x^2 + 1)^5(13x^2 + 1)$ c. $\frac{2}{(x-1)^3}$ d. $-\frac{1}{(2x-1)\sqrt{2x-1}}$ 10. a. $120x - 18$ b. $-\frac{12}{x^4}$

c. $\frac{81}{8(3x-1)^2\sqrt{3x-1}}$ d. $384x - 576$ 11. $y = 7x - 5$ 12. $-\frac{2}{25}x+\frac{27}{50}$ 13. a. $\frac{5}{2}x\sqrt{x}f(x)+x^2\sqrt{x}f'(x)$

b. $3x^2[f(x)]^2 + 2x^3f(x)f'(x)$ c. $\dfrac{3x^2 f(x)-x^3 f'(x)}{[f(x)]^2}$ d. $\dfrac{2xf(x)+f(x)+1}{2\sqrt{x}}$ 14. use the product rule twice 15. $-\dfrac{63}{25}$

16. a. $C'(t)=\dfrac{0.2-0.2t^2}{(t^2+1)^2}$ b. $0,-\dfrac{3}{125}$ 17. 6 18. $f'\left(\dfrac{g(x)h(x)}{m(n(x))}\right)\dfrac{(g'(x)h(x)+g(x)h'(x))m(n(x))-g(x)h(x)m'(n(x))n'(x)}{(m(n(x)))^2}$

19. $-3+4\sqrt{3}$ 20. 1 21. $-\dfrac{15}{4}$ 22. $(-1)^n\dfrac{1}{2}\dfrac{n!}{x^{n+1}}$

EXERCISES 1.3

1. a. $3e^x$ b. $3e^{3x-1}$ c. $2xe^{x^2-1}$ d. $-2e^{-2x}$ e. $2^x\ln 2$ f. $(\frac{1}{3})^x\ln\frac{1}{3}$ g. $e^x3^x(\ln 3+1)$ h. $e^x(x+1)$ i. $2x+2e^x$ j. $\dfrac{-2e^{2x}}{(e^x-1)^2}$ k. $\dfrac{e^x}{2\sqrt{e^x-1}}$

l. $100(e^x+x)^{99}(e^x+1)$ m. $\dfrac{-e^x}{2(e^x+1)\sqrt{e^x+1}}$ n. $2xe^x+4e^{2x}+e^x-1$ o. $\dfrac{e^x-e^{-x}}{3}$ p. $-\dfrac{e^{\frac{1}{2}x}}{2x^2}$ q. $\dfrac{e^{\sqrt{x}+1}}{2\sqrt{x}}$ r. $\dfrac{e^{\frac{\sqrt{x}-1}{\sqrt{x}+1}}}{\sqrt{x}(\sqrt{x}+1)^2}$

s. $2xe^{x^2-1}(1+x^2)$ t. $\dfrac{6^x\ln 6+3^x\ln 3+18^x\ln 2}{(3^x+1)^2}$ u. $2^{x^2+4x}\cdot\ln 2\cdot(2x+4)$ v. $\dfrac{(3\cdot 5^{3x+1}\cdot\ln 5)(x^2+e^x)-(2x+e^x)5^{3x+1}}{(x^2+e^x)^2}$

w. $\sqrt{2}^{x^2-x-1}(2x-1)\ln\sqrt{2}$ x. $2^{e^x+2}\cdot 3^x(e^x\ln 2+\ln 3)$ 2. a. $\dfrac{3}{x}$ b. $\dfrac{1}{x}$ c. $\dfrac{3}{x}$ d. $\dfrac{6}{2x+1}$ e. $\dfrac{7}{x}$ f. $\dfrac{1}{2x}$ g. $\dfrac{1}{x\ln 3}$ h. $\dfrac{1}{x\ln\frac{1}{2}}$ i. $\dfrac{\ln x+1}{\ln 10}$

j. $\dfrac{2x}{(x^2+1)\ln 2}$ k. $\dfrac{8x-6}{4x^2-6x+3}$ l. $\dfrac{2}{(1-x)(x+1)}$ m. $\dfrac{1}{x^2-1}$ n. $x(2\ln x+1)$ o. $\dfrac{1}{x\sqrt{\ln x^2}}$ p. $\dfrac{x+1}{2x\sqrt{\ln x+x}}$ q. $-\dfrac{1}{\sqrt{x}(\sqrt{x}-1)}$

r. $\dfrac{2x-1}{x^2-x}$ s. $e^x\left(\ln x+\dfrac{1}{x}\right)$ t. $\dfrac{2x-1-(x^2-x)\ln(x^2-x)}{e^x(x^2-x)}$ u. $\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{1}{x-1}$ v. $\dfrac{1}{x(x+1)\ln 10}$ w. $\dfrac{1}{\sqrt{x}(1-x)\ln 3}$

x. $3(e^x-\log_2 x^2)^2(e^x-\dfrac{2}{x\ln 2})$ y. $\dfrac{2x-\log_3 e}{2\sqrt{x^2-\log_3 e^x}}$ z. $\dfrac{3e^x(\log(1+e^x))^2}{1+e^x}$ 3. a. $(\dfrac{14}{2x-1}+\dfrac{44x^3}{x^4-3})((2x-1)^7(x^4-3)^{11})$

b. $(\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{2x}{x^2+1})(x(x+1)(x^2+1))$ c. $(\dfrac{2}{3x}+2x-\dfrac{6x^2-2}{x^3-x})\dfrac{\sqrt[3]{x^2}e^{x^2-1}}{(x^3-x)^2}$ d. $(\dfrac{3x}{4+3x^2}-\dfrac{2x}{3x^2+3})\dfrac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$

e. $\dfrac{\ln x+2}{2\sqrt{x}}x^{\sqrt{x}}$ f. $(\ln x+\dfrac{x+1}{x})(\ln x)^{x+1}$ 4. $y=2x-1$ 5. $y=2$ 6. $y=x-1$ 7. a. $3\cos(3x-5)$ b. $-2x\sin(x^2-1)$

c. $\cos x+\sin x$ d. $\sec^2x(2+\sin x)$ e. $\sin x(\sec^2x+1)$ f. $-\cos x+2\tan x+2x\sec^2x+x\sin x$

g. $-2\cos(2x^3-3x)\sin(2x^3-3x)\cdot(6x^2-3)$ h. $-6\sin^2(\ln\cos 2x)\cos(\ln\cos 2x)\tan 2x$ i. $-\sin x-e^3\csc^2x$

j. $\dfrac{20\sin x(1-\cos x)^9}{(\cos x+1)^{11}}$ k. $-\dfrac{\cos x(\cos x+\sin^2 x+1)}{\sin^2 x(\cos x+1)^2}$ l. $10(\dfrac{1-\cos x}{1+\cos x})^9\dfrac{2\sin x}{(1+\cos x)^2}$ m. $\dfrac{(2x-1)\sec^2\sqrt{x^2-x-1}}{2\sqrt{x^2-x-1}}$

n. $9x^2\cot^2(x^3-1)\csc^2(x^3-1)$ o. $3\left(\frac{\tan(2x-1)}{2+\ln x}\right)^2\cdot\frac{2x\sec^2(2x-1)\cdot(2+\ln x)-\tan(2x-1)}{x(2+\ln x)^2}$

p. $2((1+x)\cos e^x)(\cos e^x-(1+x)\sin e^x)$ q. $2(e^{\sin x+\cos x}+x\cos e^x)(e^{\sin x+\cos x}(\cos x-\sin x)+\cos e^x-xe^x\sin e^x)$

r. $\frac{2(\cos 2x-\sin 2x-1)}{(\sin 2x+1)(1-\cos 2x)}$ s. $5(x^2\sin(x-1))^4(2x\sin(x-1)+x^2\cos(x-1))$ t. $\frac{(2x^4-6x^2)\sec^2\frac{x^3}{x^2-1}\tan\frac{x^3}{x^2-1}}{(x^2-1)^2}$

u. $\frac{4\tan(\ln(2x+1))\sec^2(\ln(2x+1))}{2x+1}$ v. $-3\csc^3\frac{e^x-\ln x}{x^2-1}\cdot\cot\frac{e^x-\ln x}{x^2-1}\cdot\frac{(xe^x-1)(x^2-1)-(e^x-\ln x)2x^2}{x(x^2-1)^2}$

8. a. $y=x+1$ b. $y=-x$ 9. $\pm\frac{2\pi}{3}+2\pi k,\ k\in\mathbb{Z}$ 10. a. 0 b. $\frac{2}{3}$ c. 2 d. 0 11. 5 12. a. $\frac{5}{4}$ b. $\frac{y}{1-x}$ c. $\frac{3x^2+2x-y}{x}$ d. $\frac{6x^3+y}{x}$

e. $-\frac{2x}{3y}$ f. $-\frac{2x+5y}{5x+3y^2}$ g. $\frac{y-2xy^3}{-x+3x^2y^2}$ h. $\frac{y\sqrt{xy}-6xy}{4xy^2-x\sqrt{xy}}$ i. $\frac{1-e^{2y}}{e^{2y}+1}$ j. $\frac{y-xy^2}{x^2y+x}$ 13. a. $y=x-3$ b. $y=x$ c. $y=0$

d. $y=x-\frac{1}{e}$ 14. a. $\frac{2t-2}{3}$ b. $\frac{t^2+1}{t^2(2t+1)}$ c. $\frac{2t+3}{3t^2-2t}$ d. $(4t+6)\sqrt{t+1}$ e. $-\frac{3t\cdot\sqrt[3]{t^2}}{\sqrt{4-t^2}}$ f. $-\frac{5}{4}\cot t$ g. $-\frac{te^t}{t+1}$ 15. a. $y=x-1$

b. $y=\frac{4}{3}x-\frac{17}{3}$ 16. a. $\frac{6(t^2-t-1)}{(2t-1)^3}$ b. $\frac{6}{t^2}$ c. $-\frac{t+1}{t^2e^{2t}}$ d. 0 17. a. $\frac{1}{\sqrt{4-x^2}}$ b. $-\frac{x^2+1}{\sqrt{-x^6+3x^4-x^2}}$ c. $\frac{1-\frac{1}{\sqrt{1-x^2}}-x+\operatorname{Arcsin}x}{e^x}$

d. $-\frac{e^x}{\operatorname{Arccos}e^x\sqrt{1-e^{2x}}}$ e. $\frac{2x+1}{1+(x^2+x-1)^2}$ f. $\frac{1}{1+x^2}+\frac{x}{\sqrt{1-x^2}}$ g. $\frac{1}{\sqrt{2\cos^4x-\cos^2x}}$ 18. $y=\frac{\sqrt{3}x}{3}+\frac{\pi}{6}-\frac{\sqrt{3}}{3}$

19. a. $\sin x$ b. $-2^{99}\cos 2x$ c. $-35\sin x-x\cos x$ d. $3^{51}e^{3x+1}$ 20. a. $\frac{6y}{x^2}$ b. $-\frac{27x^4}{16y^7}-\frac{3x}{2y^3}$ c. $\frac{2xy+2y^2}{(x+2y)^3}$ d. $\frac{2\sqrt[3]{y}}{3x\sqrt[3]{x}}+\frac{2\sqrt[3]{y^2}}{3x\sqrt[3]{x^2}}$

21. $y=2x-2$ 22. a. $\frac{6}{5}$ b. $\frac{1}{\pi\cdot\sqrt[4]{2}}$ c. 5 23. $-\sec^2x$ 24. 48 25. $(-\infty,0)\setminus\{-1\}$ 26. 2 27. 2

EXERCISES 2.1

1. a. 0 b. $\frac{1}{6}$ c. $\frac{3}{2}$ d. -3 e. $\frac{1}{2}$ f. -1 g. 3 h. $-\frac{2}{5}$ i. $-\frac{1}{48}$ j. $-\frac{2\cdot\sqrt[6]{3^5}}{9}$ k. $\frac{5}{6}$ l. $\cos a$ m. 2 n. -1 o. $\ln\frac{3}{7}$ p. $-\frac{1}{4}$ q. $\frac{3}{4}$ r. 1
s. $\frac{1}{3}$ t. $\frac{1}{4}$ 2. a. 0 b. -1 c. 1 d. 1 e. 0 f. ∞ g. $\frac{1}{2}$ h. 0 i. $\frac{5}{3}$ j. $\frac{1}{3}$ 3. a. $\frac{4}{3}$ b. 1 c. 1 d. 0 e. 0 f. $-\frac{1}{\pi}$ g. 0 h. $\frac{3}{2}$

EXERCISES 2.2

1. a. increasing: $(-\infty,-2)$ and $(0,2)$; decreasing: $(-2,0)$ and $(2,\infty)$ b. increasing: $(-\infty,-3)$; decreasing: $(-3,0)$ and $(0,\infty)$
2. a. decreasing: $(-\infty,\infty)$ b. increasing: $(0,\infty)$; decreasing: $(-\infty,0)$ c. increasing: $(-\infty,2)$; decreasing: $(2,\infty)$

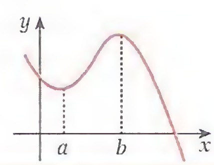
d. increasing: $(-\infty, \infty)$ e. increasing: $(-\infty, 0)$ and $(4, \infty)$; decreasing: $(0, 4)$ f. increasing: $(-2, 0)$ and $(1, \infty)$; decreasing: $(-\infty, -2)$ and $(0, 1)$ g. increasing: $(-\infty, 2)$ and $(2, \infty)$ h. increasing: $(0, \infty)$; decreasing: $(-\infty, 0)$ i. increasing: $(-\infty, 0)$ and $(10, \infty)$; decreasing: $(0, 10)$ j. increasing: $(-\infty, \infty)$ k. decreasing: $(-\infty, 5)$ and $(5, \infty)$ l. increasing: $(2, \infty)$; decreasing: $(-\infty, 2)$ m. increasing: $(0, \sqrt{e})$; decreasing: $(\sqrt{e}, \infty)$ n. increasing: $\left(-\frac{5\pi}{6}+2\pi k, \frac{\pi}{6}+2\pi k\right), k \in \mathbb{Z}$; decreasing: $\left(\frac{\pi}{6}+2\pi k; \frac{7\pi}{6}+2\pi k\right), k \in \mathbb{Z}$ k. increasing: $(-\infty, \log_{0.5}3)$ and $(0, \infty)$; decreasing: $(\log_{0.5}3, 0)$ 3. $(\frac{\pi}{4}, \frac{5\pi}{4})$

4. $f'(x) = \dfrac{1}{1+x^2} > 0$ for all x, so f is increasing 5. $-2 < m < 2$ 6. $1 < a < 4$ 7. $1 < a < 3$ 8. -8 9. a. decreasing b. increasing c. decreasing d. increasing 10. a. decreasing b. increasing c. increasing 11. local max.: x_2, x_4, x_6; local min.: x_3, x_5, x_7, x_8; absolute max.: x_6; absolute min.: x_1, 12. a. no critical point b. 0, -2 c. -1 d. -1 13. a. $f'(a) = f'(b) = f'(c) = 0$, $f'(d)$ and $f'(e)$ do not exist. b. local max.: a, d; local min.: b; neither: c, e 14. a. min.: $x = -4$ b. max.: $x = 1$; min.: $x = -1$ c. no local extrema d. min.: $x = -3$, $x = 1$; max.: $x = -1$ e. no local extrema f. min.: $x = -2$ g. max.: $x = 0$; min.: $x = -2$, $x = 2$ h. min.: $x = -3$ i. min.: $x = 1$ j. min.: $x = \frac{7\pi}{6} + 2\pi k$, $x = \frac{11\pi}{6} + 2\pi k$; max.: $x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ 15. 6 16. $a = 2$, $b = -4$ 17. $k = 4$ 18. $a = -\frac{3}{2}$, $b = -6$ 19. $-2e$ 20. $m = -3$ 21. $ab < 0$ 22. $-1 < m < 0$ 23. max.: $x = -2$, $x = 7$; min.: $x = 3$ 24. a. $f_{\min} = 1, f_{\max} = 3$ b. $f_{\min} = -9, f_{\max} = 0$ c. $f_{\min} = -88, f_{\max} = -7$ d. $f_{\min} = 19, f_{\max} = 30$ e. $f_{\min} = -3, f_{\max} = 9 - 4\sqrt{3}$

f. $f_{\min} = -255, f_{\max} = 1$ g. $f_{\min} = 0, f_{\max} = 20$ h. $f_{\min} = \sqrt{5}, f_{\max} = 3$ i. $f_{\min} = 0, f_{\max} = \frac{8}{3}$ j. $f_{\min} = 0, f_{\max} = 3$ k. $f_{\min} = -1, f_{\max} = \frac{3}{5}$ l. $f_{\min} = 0, f_{\max} = 4$ m. $f_{\min} = 1, f_{\max} = \frac{23}{3}$ n. $f_{\min} = -7, f_{\max} = 3$ 25. 24 26. 2 27. $a = -\frac{1}{4}$, $b = -3$

28. $\sqrt{2}$ 29. a. increasing: $(-2, \infty)$; decreasing: $(-\infty, -2)$ b. min.: $x = -2$ 30. a. increasing: $(-\infty, -4)$ and $(4, 6)$; decreasing: $(-4, 4)$ and $(6, \infty)$; b. min.: $x = 4$; max.: $x = -4$, $x = 6$ 31. $x = 2$ 33. $(-\infty, 6] \cup [16, \infty)$ 34. $a \geq 3$

32.

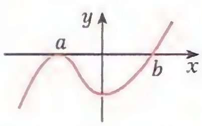


EXERCISES 2.3

1. a. positive: $(0, \infty)$; negative: $(-\infty, 0)$ b. positive: $(-\infty, -3)$, $(-3, -1)$, $(2, \infty)$; negative: $(-1, 2)$ 2. a. concave up: $(-\infty, \infty)$, no inflection point b. concave down: $(-\infty, \infty)$, no inflection point c. concave up: $(-\frac{1}{3}, \infty)$, concave down: $(-\infty, -\frac{1}{3})$, inflection point $x = -\frac{1}{3}$ d. concave up: $(1, \infty)$, concave down: $(-\infty, 1)$, inflection point: $x = 1$ e. concave up: $(-\infty, 1)$, $(\frac{5}{3}, \infty)$, concave down: $(1, \frac{5}{3})$, inflection points: $x = 1$ and $x = \frac{5}{3}$ f. concave up: $(-\infty, -2)$, $(2, \infty)$, concave down: $(-2, 2)$ inflection points: $x = -2$ and $x = 2$ g. concave up: $(-1, 0)$, $(1, \infty)$, concave down: $(-\infty, -1)$, $(0, 1)$ inflection points: $x = -1$, $x = 0$, $x = 1$. h. concave up: $(-\infty, 0)$, $(0, \infty)$, no inflection point

i. concave down: $(-1, \infty)$, no inflection point j. concave up: $(5, \infty)$, concave down: $(-\infty, 5)$, inflection point: $x = 5$ k. concave up: $(0, \infty)$, concave down: $(-\infty, 0)$, inflection point $x = 0$ l. concave up: $(1, 2)$, concave down: $(2, 3)$, inflection point $x = 2$ m. concave up: $(2\pi k, \pi + 2\pi k)$, concave down: $(\pi + 2\pi k, 2\pi + 2\pi k)$, inflection point $x = \pi + \pi k$, $k \in \mathbb{Z}$ 3. $a = 6$, $b = 0$ 4. $(-1, -19)$ 5. $(1, \frac{e}{2})$ 6. $(\frac{\pi}{2}, e^{\pi/2})$ 7. concave up: $(-1, 3)$, concave down: $(-\infty, -1)$, $(3, \infty)$, inflection points: $x = -1$, $x = 3$ 8. a. minimum b. maximum c. minimum d. minimum

9. a. max.: $x = -3$, min.: $x = -1$ b. max : $x = \frac{-3+\sqrt{5}}{2}$, min : $x = 0$, $x = \frac{-3-\sqrt{5}}{2}$ c. max.: $x = 0$, min.: $x = 4$ d. min.: $x = -3$ e. max.: $x = 0$ f. no extremum

10. 

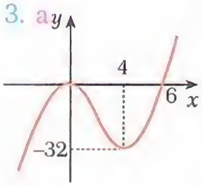
11. consider the second derivative of $f(x) = ax^3 + bx^2 + cx + d$ 12. consider the second derivative of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ 13. $y = -x + 3$ 14. $a = -1$, $b = 0$, $c = 3$ 15. $c = 2$ 16. $2f'(a) - f(a)$

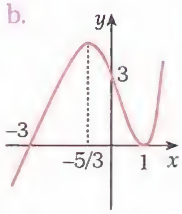
EXERCISES 2.4

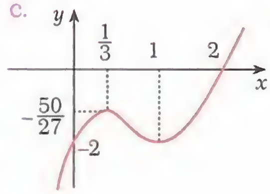
1. 15 and 15 2. -2 and 2 3. 50 4. $2\sqrt{m}$ 5. $m = 1$ 6. 50 m by 50 m 7. 30 m by 60 m 8. $\frac{512}{3}$ cm^3 9. 2, 2 and 1 m 10. 3888 cm^3 11. $\frac{6}{\sqrt[3]{\pi}}$ m 12. 250 13. $2R^2$ 14. $2\sqrt{2}$ 15. $(1, 0)$ 16. 24 17. $y = 2\sqrt{2}$ 18. 13 19. $r = 2$ cm, $h = \frac{5}{3}$ cm 20. 1 hour 21. 9

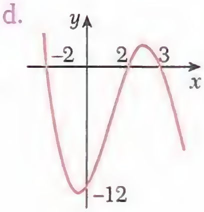
EXERCISES 2.5

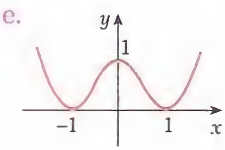
1. a. $x = -1$, $y = 0$ b. $x = 1$, $y = 0$ c. no asymptotes d. $x = 4$, $y = -3$ e. $x = -3$, $y = 0$ f. $x = \pm 1$, $y = -1$ g. $x = -3$, $y = 1$ h. $x = \frac{3}{2}$, $x = -3$, $y = -\frac{3}{2}$ i. $x = 6$, $x = -1$, $y = 0$ j. $y = x$ k. $x = 1$, $y = x + 3$ 2. $-\frac{4}{9}$, $\frac{1}{4}$

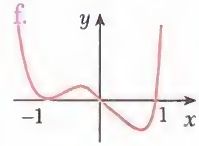
3. a. 

b. 

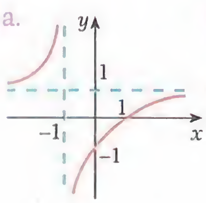
c. 

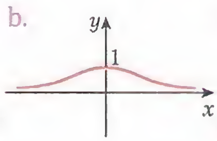
d. 

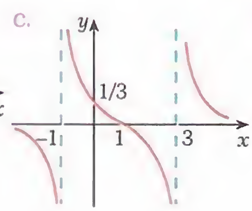
e. 

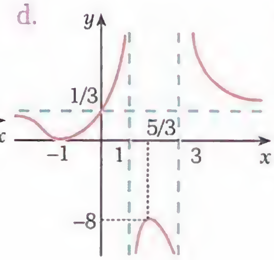
f. 

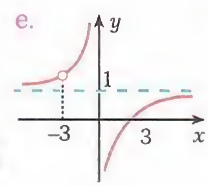
4. $-\frac{4}{27}$ 5. $a = -\frac{3}{14}$, $b = -\frac{7}{2}$ 6. 0 7. $y = \frac{1}{2}(x-2)^2(x+1)(x+2)$

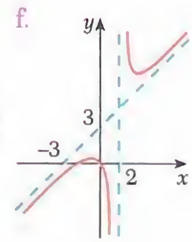
8 a. 

b. 

c. 

d. 

e. 

f. 

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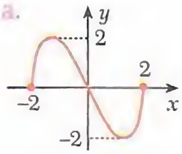

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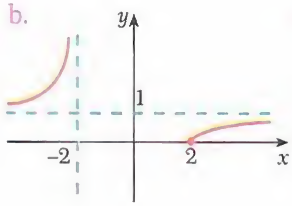

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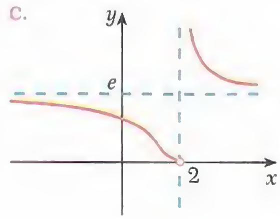

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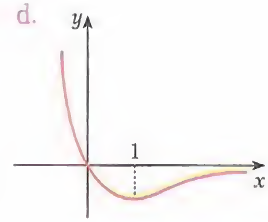

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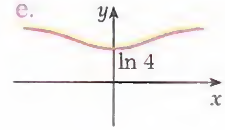

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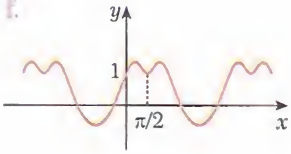

10. 1 solution if $a \in (-\infty, -4) \cup (23, \infty)$, 2 solutions if $a \in \{-4, 23\}$, 3 solutions if $a \in (-4, 23)$

ANSWERS TO TESTS

TEST 1A

1.	C	9.	E
2.	D	10.	C
3.	A	11.	A
4.	E	12.	B
5.	C	13.	A
6.	D	14.	D
7.	C	15.	C
8.	A	16.	D

TEST 1B

1.	C	9.	A
2.	D	10.	E
3.	C	11.	E
4.	B	12.	A
5.	B	13.	E
6.	A	14.	C
7.	A	15.	A
8.	E	16.	B

TEST 1C

1.	E	9.	D
2.	C	10.	E
3.	E	11.	B
4.	B	12.	D
5.	C	13.	B
6.	D	14.	D
7.	C	15.	A
8.	E	16.	B

TEST 2A

1.	B	9.	B
2.	C	10.	C
3.	A	11.	C
4.	D	12.	D
5.	E	13.	B
6.	E	14.	A
7.	D	15.	D
8.	A	16.	E

TEST 2B

1.	E	9.	E
2.	B	10.	D
3.	E	11.	C
4.	D	12.	E
5.	B	13.	A
6.	A	14.	C
7.	A	15.	D
8.	D	16.	B

TEST 2C

1.	B	9.	E
2.	E	10.	E
3.	E	11.	D
4.	B	12.	B
5.	A	13.	E
6.	D	14.	A
7.	D	15.	E
8.	D	16.	A

GLOSSARY

A

absolute maximum: the greatest value of a function on its domain.
absolute minimum: the smallest value of a function on its domain.
asymptote: a line that a curve approaches more and more closely until the distance between the curve and the line almost vanishes.

C

concave down: (of a curve on an interval) having a decreasing derivative as the independent variable increases.
concave up: (of a curve on an interval) having an increasing derivative as the independent variable increases.
constant function: a function whose image set is of a unique element.
continuity at a point: equality of the value of a function and the limit of the function at a point.
continuous function: a function whose graph has no breaks.
critical point: a point on the graph of f where $f'(x) = 0$ or $f'(x)$ does not exist.

D

decreasing function: a function whose values decrease as the independent variable increases, or vice-versa.
differentiable function: a function which has a derivative at a given point.
differentiation: finding the derivative of a function.
discontinuous function: a function whose graph has a break.
domain of a function: the set of real numbers for which the function is defined.
double root: one of a pair of equal roots of the same polynomial or equation.

E

extremum of a function: maximum or minimum value of the function.

I

implicit function: a function defined by an equation of the form $f(x, y) = 0$.
increasing function: a function whose values increase as the independent variable increases, or vice-versa.
indeterminate form: a form for which function is not defined in real numbers even the limit exists.
inflection point: a point on the graph of f where $f''(x) = 0$ or $f''(x)$ does not exist.
interval: a set containing all real numbers, called the endpoints. A closed interval includes the endpoints, but an open interval does not.

L

left-hand derivative: the limit of a difference quotient $\dfrac{f(x+h)-f(x)}{h}$ as h approaches zero from the left.
l'Hospital's rule: a rule permitting the evaluation of the limit of an indeterminate quotient of functions.
local maximum: the greatest value of a function on an interval.
local minimum: the smallest value of a function on an interval.

N

normal line to a curve: a line perpendicular to a tangent of the curve at the point of tangency.

P

parabola: the graph of a quadratic function.
parametric function: a function in which coordinates are each expressed in terms of parameters.
polynomial function: a function of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a positive integer and $a_n \neq 0$.

Q

quotient: the ratio of two numbers or quantities.

R

rational function: a function of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
right-hand derivative: the limit of a difference quotient $\dfrac{f(x+h)-f(x)}{h}$ as h approaches zero from the right.

S

secant line to a curve: a line that intersects the curve at two or more points.
slope of a line: a number measuring the steepness of the line relative to the x-axis.

T

tangent line to a curve: a line that touches the curve at exactly one point.

X

x-intercept: the value of x at a point where a graph crosses the x-axis.

Y

y-intercept: the value of y at a point where a graph crosses the y-axis.

DERIVATIVES

MODULAR SYSTEM

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